THE NONLINEAR UNSYMMETRICALLY LAMINATED COMPOSITE BEAMS ON WINKLER – PASTERNAK FOUNDATION

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*Abstract***.**In this study, Optimal Parametric Iteration Method (OPIM) is presented for large amplitude free vibrations of nonlinear unsymmetrically laminated composite beams on Winkler – Pasternak elastic foundation. Based on von Kárman geometric nonlinearity, on Euler – Bernoulli beam theory and Galerkin procedure, we obtain a second-order nonlinear differential equation with quadratic and cubic terms. Comparison between results of the present work and those of numerical results shows the accuracy of our approach.

Keywords: Nonlinear equations, composite beam, OPIM.

1. INTRODUCTION

This paper is devoted to laminated composite beams with high stiffness and strength to weight ratio. These are increasingly used in many various fields of engineering like civil and aerospace engineering, automotive, semiconductor industry, biomedical science, defense industry.

Natural responses of these composite structures are assentially nonlinear and are described by nonlinear differential equations. It is known that a nonlinear problem is oftem difficult to find an exact solution. Many researchers have investigated different aspects of composite beams or functionally graded beams. Dokmeci [1] presented the system of one-dimensional equations to analyze the thermoviscoelastic behavior of an axially functionally graded beams of rectangular cross section at high-frequency vibration. Also, the vibrations of simply supported functionally graded beams with piezoelectric layers subjected to axial compressive loads is studied by Khorramabadi [2].

Younesian and Esmailzadeh [3] analyzed the coupled longitudinal and bending vibration of a beam and the governing equations of motions, using Hamilton's principle are derived. The multiple seales method is then utilized to obtain the first order approximate solution. The approximate analytical expression for geometricall nonlinear vibration and post-buckling analysis of functionally graded beams on nonlinear foundation is obtained by Fallah and Aghdam [4], by

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using variational iteration method. The same procedure is applied by Baghani et al. [5] to obtain a closed-form solution for the large amplitude free vibrations and post-buckling analysis of unsymmetrical laminates composite thin beams.

Challamel and Girhammar [6] investigated the stability problem of a partial composite column, or equivalent composite sandwich column with and whitout the effect of shear, with or whitout the effect of axial extensibility and also the effect of eccentric axial load application. By using a variational formulation, the buckling behavior of partial composite columns is analyzed with respect to both the Engesser and the Harings theory. Emam [7] examinated the significance of the shear deformation of the static postbucking response of composite beams. An exact solution for the static postbuckling response of a symmetrically laminated simply supported shear-deformable composite beams is obtained.

The Wittrick-Williams algorithm is used by Su and Benerjee [8] as solution technique to yield natural frequencies and mode shapes of the functionally graded beams. The analytical expression for axial force, shear force and bending moments at any cross-section are obtained as a by-product of the Hamilton formulation. Thomas et al. [9] considered the finite element modelling and free vibratio analysis of functionally graded nanocomposite beams reinforced by randomly oriented straight single-walled carbon nanotubes. The Eshelby – Mori – Tanaka approach based on an equivalent fiber is used to investigate the material properties of the beam.

The nonlinear vibration and post-buckling analysis of beams made of functionally graded beam rest on a non-linear elastic foundation subiect to an axial force is studied by Yaghoobi and Torabi [10] by means of variational iteration method. Akbaş [11] explored free vibration and static bending behavior of simple supported functionally graded beams resting on Winkler foundation with Euler – Bernoulli and Timoshenko beam theory. The Navier-type method is used and the effects of foundation stiffness and different material distributions on the natural frequencies and the bending responses are investigated. Elmaguiri et al. [12] employed the large-amplitude free vibration of champed immovable thin beams made of functionally graded materials by applying Lagrange’s equations and the harmonie balance method.

The free vibration of a rotating beam composed of functionally graded materials based on Mori – Tanaka model is analyzed in conjunction withTimoshenko beam theory by Ebrahimi and Mokhtari [13]. The physical neutral axis position for this type of beam is determined and the governing differential equations one solved by a semi-analytical differential transformation method. Vibrations of non-uniform and functionally graded beams with various boundary conditions and varying cross-society are investigated used the Euler – Bernoulli theory and Haar matrices. The Haar wavelet approach for calculating natural frequences on non-uniform beams is used.

The objective of this study is to use the OPIM to obtain a closed-form solution for the large amplitude free vibrations and post-buckling analysis of unsymmetrical laminated composite thin beams on the Winkler – Pasternak foundation. Considering von Kárman geometrie nonlinearity, the Euler – Bernoulli beam theory and Galerkin procedure, we obtain a second-order nonlinear differential equation with quadratic and cubic nonlinear terms. The accuracy of the analytical results obtained through the proposed approach is proved by numerical simulations developed in order to validate analytical results. A numerical example showes that the proposed approach is simple and easy to use.

2. THE GOVERNING EQUATION OF MOTION

In what follows, we consider a straight laminated beam of length L, width b and thickness h resting on an elastic foundation of Winkler – Pasternak type and subjected to an axial compressive force P (Fig. 1). The material properties are considered to vary in accordance with the rule.

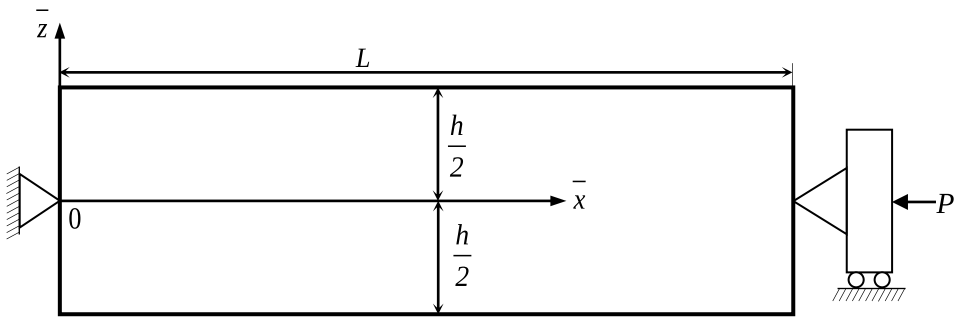

Fig. 1 – The laminar composite beams simply supported end conditions.

$$P = P_B V_B + P_T V_T \,, \tag{1}$$

in which P and V are material property and volume fraction, respectively and the subscripts B and T refer to two different materials. In this work, we consider a simple power law to describe variation of material properties from to bottom face $\left(\bar{z} = -\frac{h}{2}\right)$ to the top face $\left(\bar{z} = \frac{h}{2}\right)$ of the beam as [3], [4].

$$V_T = \left(\frac{2\bar{z} + h}{2h}\right)^k ; \quad V_B = 1 - V_T \tag{2}$$

where k characterises the distribution of material properties. The case $k = 0$ corresponds to an isotropic homogeneous beam.

Taking into consideration to rule of mixtures, the modalus of elasticity (E), Poisson's ratio (ν) and mass density (ρ), we have:

$$E(\bar{z}) = E_B + (E_T - E_B)\left(\frac{2\bar{z}+h}{2h}\right)^k \tag{3}$$

$$\nu(\bar{z}) = \nu_B + (\nu_T - \nu_B)\left(\frac{2\bar{z}+h}{2h}\right)^k \tag{4}$$

$$\rho(\bar{z}) = \rho_B + (\rho_T - \rho_B)\left(\frac{2\bar{z}+h}{2h}\right)^k \tag{5}$$

For a small strain, moderate deformation and rotation, the axial strain of the midplane of the beam accounting for the midplane stretching is given by [15]:

$$\varepsilon_x = \frac{\partial \bar{U}}{\partial x} + \bar{z}\frac{\partial^2 \bar{W}}{\partial \bar{x}^2} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \bar{x}}\right)^2 \tag{6}$$

where, based on Euler-Bernoulli beam theory, the displacement of an arbitrary point along the $\bar{x}$ and $\bar{z}$ axes are $\bar{U}(\bar{x},\bar{z},\bar{t})$ and $\bar{W}(\bar{x},\bar{z},\bar{t})$ respectively. If $\bar{U}(\bar{x},\bar{t})$ and $\bar{W}(\bar{x},\bar{t})$ are displacement components in the midplane, then we have:

$$\bar{U}(\bar{x},\bar{z},\bar{t}) = \bar{U}(\bar{x},\bar{t}) + \bar{z}\frac{\partial \overline{W}}{\partial \bar{x}}, \bar{W}(\bar{x},\bar{z},\bar{t}) = \bar{W}(\bar{x},\bar{t}) \tag{7}$$

where $\bar{t}$ is the time. The normal stress for the von Kárman type of geometry is given by the law:

$$\delta_{xx} = \frac{E(\bar{z})}{1-\nu^2(\bar{z})}\left[\frac{\partial \bar{U}}{\partial \bar{x}} + \bar{z}\frac{\partial^2 \bar{W}}{\partial \bar{x}^2} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \bar{x}}\right)^2\right]. \tag{8}$$

The curvature of the beam is given by

$$\frac{1}{R} = \frac{\partial^2 \bar{W}}{\partial \bar{x}^2}. \tag{9}$$

The axial, coupling and bending stifness are defined as:

$$\left(A_{11}, B_{11}, C_{11}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(\bar{z})}{1-\nu^2(\bar{z})}\left(1, \bar{z}, \bar{z}^2\right) d\bar{z} \tag{10}$$

The total induced axial force N_x and bending moment M_x are related to the stress resultants as:

$$N_x = A_{11}\left[\frac{\partial \bar{U}}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \bar{x}}\right)^2\right] + B_{11}\frac{\partial^2 \bar{W}}{\partial \bar{x}^2}, \tag{11}$$

$$M_x = B_{11}\left[\frac{\partial \bar{U}}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \bar{x}}\right)^2\right] + D_{11}\frac{\partial^2 \bar{W}}{\partial \bar{x}^2}. \tag{12}$$

The equations of motion for axial and transverse vibration of laminated composite beam based on Euler-Bernoulli beam theory and von Kárman geometric nonlinearity are:

$$I_1\frac{\partial \bar{U}}{\partial \bar{z}} - \frac{\partial N_x}{\partial \bar{x}} = 0, \tag{13}$$

$$\frac{\partial^2 M_x}{\partial \bar{x}^2} - \frac{\partial}{\partial \bar{x}}\left(N_x\frac{\partial \bar{W}}{\partial \bar{x}}\right) + I_1\frac{\partial^2 \bar{W}}{\partial \bar{t}^2} = F_w, \tag{14}$$

when inertia term and reaction of the elastic foundation of the beam are:

$$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\bar{z}) d\bar{z}, \quad F_W = -k_L W - k_{NL} W^3 + k_s\frac{\partial^2 W}{\partial \bar{z}^2}, \tag{15}$$

in which k_L and k_{NL} are the linear and non-linear coefficients respectively and k_s is the shear coefficient of elastic foundation.

Neglecting the axial inertia, from Eq. (11), follows that N_x is independent of $\bar{x}$:

$$N_x = N_{x0} = A_{11}\left[\frac{\partial \bar{U}}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \bar{x}}\right)^2\right] + B_{11}\frac{\partial^2 \bar{W}}{\partial \bar{x}^2} \tag{16}$$

and therefore, from Eq. (16), we obtain:

$$A_{11}^{-1}\left(N_x-B_{11}\frac{\partial^2\bar{W}}{\partial\bar{x}^2}\right)-\frac{1}{2}\left(\frac{\partial\bar{W}}{\partial\bar{x}}\right)=\frac{\partial\bar{U}}{\partial\bar{x}}\,. \tag{17}$$

Supposing that the beam has immovable ends:

$$\bar{U}\left(0,\bar{t}\right)=\bar{U}\left(L,\bar{t}\right)=0 \tag{18}$$

and integrating Eq. (17) with respect to $\bar{x}$, and taking into account that N_x is independent of $\bar{x}$, one can get:

$$N_x=\frac{A_{11}}{2L}\int_0^L\bar{W}'^2d\bar{x}+\frac{B_{11}}{L}\left[\bar{W}'\left[L,\bar{t}\right]-\bar{W}'\left(0,\bar{t}\right)\right]-\bar{P}\,. \tag{19}$$

From Eq. (11) and (19) results

$$\frac{\partial\bar{U}}{\partial\bar{x}}=A_{11}^{-1}N_x-\frac{1}{2}\bar{W}'^2-B_{11}\bar{W}' \tag{20}$$

or

$$\frac{\partial\bar{U}}{\partial\bar{x}}=\frac{1}{2L}\int_0^L\bar{W}'^2d\bar{x}+\frac{B_{11}}{A_{11}L}\left[\bar{W}'\left(L,\bar{t}\right)-\bar{W}'\left(0,\bar{t}\right)\right]-\frac{1}{2}\bar{W}^2-\frac{B_{11}}{A_{11}}\bar{W}''. \tag{21}$$

Differentiating the last equation with respect to $\bar{x}$, yields:

$$\frac{\partial^2\bar{U}}{\partial\bar{x}^2}=-\bar{W}'\bar{W}''-\frac{B_{11}}{A_{11}}\bar{W}''', \tag{22}$$

where prime denotes derivative with respect to $\bar{x}$.

From Eqs (12) and (21), we have that:

$$M_x=\frac{B_{11}}{2L}\int_0^L\bar{W}'^2d\bar{x}+\frac{B_{11}^2}{A_{11}L}\left[\bar{W}'\left(L,\bar{t}\right)-\bar{W}'\left(0,\bar{t}\right)\right]-\frac{B_{11}}{A_{11}}\bar{W}''+D_{11}\bar{W}. \tag{23}$$

Furthermore, using Eqs. (12) and (23) it holds that:

$$\frac{\partial^2M_x}{\partial\bar{x}^2}=\left(D_{11}-\frac{B_{11}^2}{A_{11}L}\right)\bar{W}^{(IV)}\,. \tag{24}$$

From Eqs. (19), (24) and (14), we obtain the governing nonlinear laminated composite vibration equation as follows:

$$
\begin{aligned}
& I_1\ddot{W} + \left(D_{11} - \frac{B_{11}^2}{A_{11}L}\right)\overline{W}^{(IV)} - \\
& -\overline{W}''\left[\frac{A_{11}}{2L}\int_0^L \overline{W}^2 d\overline{x} + \frac{B_{11}}{L}\left(\overline{W}'\left(L,\overline{t}\right) - \overline{W}'\left(0,\overline{t}\right) - \overline{P}\right)\right] = F_W,
\end{aligned} \tag{25}
$$

where the dot denotes derivative with respect to time.

For the simplicity of the parametric studies, the following dimensionless variables are introduced

$$
\begin{aligned}
& x = \frac{\overline{x}}{L};\, W = \frac{\overline{W}}{h};\, r = \sqrt{\frac{I_1}{A}};\, 2\alpha = A_{11}r^2\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)^{-1}; \\
& t = \frac{\overline{t}}{L}\sqrt{\left(D_{11} - \frac{B_{11}^2}{A_{11}L}\right)D_{11}^{-1}}; \\
& P = \overline{P}L^2\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)^{-1};\, \lambda = B_{11}r^2\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)^{-1}; \\
& k_1 = k_L L^4\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)^{-1};\, k_2 = k_{NL}r^2L^4\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)^{-1}; \\
& k_3 = k_s L^2\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)^{-1} + \alpha\overline{P}.
\end{aligned} \tag{26}
$$

From Eqs. (15), (25 and (26) it results:

$$
\begin{aligned}
& \ddot{W} + W^{(IV)} + k_1W + k_2W^3 - k_3W'' - \\
& -\alpha W''\left[\int_0^1 W'^2 dx + \lambda\left(W'(1,t) - W'(0,t)\right)\right] = 0.
\end{aligned} \tag{27}
$$

The Eq. (27) is a partial differential equation in two dimensions: displacement x and the time t. Using Galerkin procedure, Eq. (27) becomes an ordinary differential equation. Assuming that the transverse displacement is expressed in the form:

$$
W(x,t) = X(x)T(t), \tag{28}
$$

where $X(x)$ is the linear fundamental vibration mode and $T(t)$ is the time dependent function to be determined. Substituting Eq. (28) into Eq. (27) and applying Galerkin's method, in which to orthogonality property of the mode shapes is used, yields:

$$\ddot{T}+\alpha_1 T+\alpha_2 T^2+\alpha_3 T^3=0, \tag{29}$$

where

$$\begin{aligned}
\alpha_1 &= \int_0^1 X(x)X^{IV}(x)dx+k_1\int_0^1 X^2(x)dx-k_3\int_0^1 X(x)X''(x)dx,\\
\alpha_2 &= -\alpha\lambda\left[X'(1)-X'(0)\right]\int_0^1 X(x)X''(x)dx,\\
\alpha_3 &= k_2\int_0^1 X^{IV}(x)dx-\alpha\left(\int_0^1 X'^2(x)dx\right)\left(\int_0^1 X(x)X''(x)dx\right).
\end{aligned} \tag{30}$$

It should be emphasised that for isotropic and symmetrically laminated composite beams, the coefficient α_2 is identiffically zero. Our analysis of nonlinear vibrations for unsymmetrically laminated beams is significantly different from that of isotopic and symmetrically laminated beams.

For the case of simply supported beam, the fundamental vibration made is:

$$X(x)=\sqrt{2}\sin\pi x \tag{31}$$

The Eq. (29) contains a quadratic nonlinear term due to the presence of bending-extension coupling effect and cubic nonlinear term due to the Winkler-Pasternak foundation. This equation does not have an exact solution, but by means of OPIM may be obtained an approximate solution using a set of optimal convergence-control parameters C_i. We remark that in Eq. (29) there exist no small or large parameter.

The laminated composite beam is subjected to the following initial conditions:

$$T(0)=A,\quad T(0)=0. \tag{32}$$

3. BASIC IDEA OF OPTIMAL PARAMETRIC ITERATION METHOD

We consider the general nonlinear differential equation [16], [17]:

$$L\left(T(t)\right)+N\left(t,T(t),\dot{T}(t),\ddot{T}(t)\right)=0, \tag{33}$$

with L a linear operator and N a nonlinear operator.

The initial conditions are:

$$B\left(T,\dot{T}\right)=0. \tag{34}$$

If α,β,γ are three real values, the applying the well-known Taylor formula for an arbitrary analytic function $F\left(t,T,\dot{T},\ddot{T}\right)$, we obtain:

$$\begin{aligned}&F\left(t,T+\alpha,\dot{T}+\beta,\ddot{T}+\gamma\right)=F\left(t,T,\dot{T},\ddot{T}\right)+\frac{\alpha}{1!}F_T\left(t,T,\dot{T},\ddot{T}\right)+\\&\frac{\beta}{1!}F_{\dot{T}}\left(t,T,\dot{T},\ddot{T}\right)+\frac{\gamma}{1!}F_{\ddot{T}}\left(t,T,\dot{T},\ddot{T}\right)+\ldots\end{aligned} \tag{35}$$

where $F_T=\dfrac{\partial F}{\partial T}$.

Instead of solving the nonlinear defferential Eq. (33), one can solve another equation, making recourse to Eq. (35) and the following scheme:

$$\begin{aligned}&L\left(T_{n+1}(t)\right)+N\left(t,T_n(t),\dot{T}_n(t),\ddot{T}_n(t)\right)+\\&+\alpha_n\left(t,C_i\right)N_T\left(t,T_n(t),\dot{T}_n(t),\ddot{T}_n(t)\right)+\\&+\beta_n\left(t,C_j\right)N_{\dot{T}}\left(t,T_n,\dot{T}_n,\ddot{T}_n\right)+\gamma_n\left(t,C_k\right)N_{\ddot{T}}\left(t,T_n,\dot{T}_n,\ddot{T}_n\right)=0,\\&n=0,1,2,\ldots\end{aligned} \tag{36}$$

with the initial conditions:

$$B\left(T_n,\dot{T}_n\right)=0, \tag{37}$$

where α_n,β_n and γ_n are so-called axiliary functions. The initial approximation $T_0(t)$ can be determined at least in the following two alternatives.

In the first alternative $T_0(t)$ is the solution of the linear equation:

$$L\left(T_0(t)\right)=0,\quad B\left(T_0,\dot{T}_0\right)=0. \tag{38}$$

In the second alternative we can choose the initial approximation in the form:

$$T_0(t)=\sum_{i=1}^{m}C_i f_i(t), \tag{39}$$

where C_i are m unknown constant, the functions f_i depend on the form of linear operator L and also on the coefficients of Eq. (33), and m is an integer positive number.

For both alternatives, the auxiliary functions $\alpha_n(t,C_i)$, $\beta_n(t,C_j)$ and $\gamma_n(t,C_k)$ can be chosen such that the products $\alpha_n N_T$, $\beta_n N_{\dot{T}}$ and $\gamma_n N_{\ddot{T}}$ be of the same form, respectively. The auxiliary functions f_i, α_n, β_n and γ_n are not unique. In this way, we obtain the approximation of the solution of Eq. (33) given by a truncated series (36). To improve the order of convergence of the sequence $T_n(n)$ as given in eq. (36) we propose that the constants C_1, C_2 which appear into Eqs. (36) and/or Eq. (39) and are unknown at this moment, to be determined optimally using the least square method, the Galerkin method, the Ritz method, the collocation method, and so on.

In this way, the solution of Eq. (33) is well-determined after only one iteration $(n=0)$. The basic ideas of the proposed procedure are the construction of a new iteration scheme (36) and the involvement of the convergence-control parameters $C_i, i=1,2,...$ through the auxiliary functions α_n, β_n and γ_n which lead to an excellent agrement between approximate solutions and exact solution of Eq. (33). The presence of a finite number of initially unknown parameters C_i which are optimally determined later, provides a rigorous way to control the convergence of the approximate analytic solution.

In what foloows we apply OPIM to nonlinear unsymmetrically laminated composite beam given by eqs. (29) and (32).

4. APPLICATION OF OPIM TO EQS. (29) AND (32)

Under the transformations:

$$\tau = \Omega t, T(t) = Ax(\tau) \tag{40}$$

equations (29) and (32) become:

$$x''(\tau) + \frac{\alpha_1}{\Omega^2} x(\tau) + \frac{\alpha_2 A}{\Omega^2} x^2(\tau) + \frac{\alpha_3 A^2}{\Omega^2} x^3(\tau) = 0, \tag{41}$$

respectively

$$x(0) = 1, \quad x'(0) = 0, \tag{42}$$

where prime denotes differentation with respect to τ.

In this case, the operators of Eqs. (41) and (42) are respectively:

$$L\left[x(\tau)\right]=x''(\tau)+x(\tau), \tag{43}$$

$$N\left[x(\tau)\right]=\left(\frac{\alpha_1}{\Omega^2}-1\right)x(\tau)+\frac{\alpha_2 A}{\Omega^2}x^2(\tau)+\frac{\alpha_3 A^2}{\Omega^2}x^3(\tau). \tag{44}$$

From Eq. (44), follows that:

$$N_x\left[x(\tau)\right]=\frac{\partial N}{\partial x}=\frac{\alpha_1}{\Omega^2}-1+\frac{2\alpha_2 A}{\Omega^2}x+\frac{3\alpha_3 A^2}{\Omega^2}x^2. \tag{45}$$

Applying the first alternative, the initial approximation $x_0(\tau)$ is obtained from Eq. (38):

$$x_0''(\tau)+x_0(\tau)=0,\quad x_0(0)=1,\quad x_0'(0)=0 \tag{46}$$

whose solution is:

$$x_0(\tau)=\cos\tau. \tag{47}$$

The first approximation $x_1(\tau)$ is obtained from Eq. (36):

$$x_1''(\tau)+x_1(\tau)+N\left(x_0(\tau)\right)+\alpha(\tau,C_1,C_2,...)N_x\left(x_0(\tau)\right)=0. \tag{48}$$

Substituting Eq. (47) into Eq. (45), it holds that:

$$\begin{aligned}N\left[x_0(\tau)\right]&=N(\cos\tau)=\left(\frac{\alpha_1}{\Omega^2}-1\right)\cos\tau+\frac{\alpha_2 A}{\Omega^2}\cos^2\tau+\frac{\alpha_3 A^2}{\Omega^2}\cos^3\tau\\ N_x\left[x_0(\tau)\right]&=\frac{\alpha_1}{\Omega^2}-1+\frac{2\alpha_2 A}{\Omega^2}\cos\tau+\frac{3\alpha_3 A^2}{\Omega^2}\cos^2\tau.\end{aligned} \tag{49}$$

After simple exhibitions, Eq. (49) can be rewritten in the forms:

$$N\left[x_0(\tau)\right]=M_0+M_1\cos\tau+M_2\cos 2\tau, \tag{50}$$

$$N_x\left[x_0(\tau)\right]=P_0+P_1\cos\tau+P_2\cos 2\tau, \tag{51}$$

where

$$M_0=\frac{\alpha_2 A^2}{2\Omega^2};\ M_1=\frac{\alpha_1}{\Omega^2}+\frac{3\alpha_3 A^2}{4\Omega^2}-1;\ M_2=\frac{\alpha_2 A^2}{2\Omega^2}; \tag{52}$$

$$P_0=\frac{\alpha_1}{\Omega^2}+\frac{3\alpha_3A^2}{2\Omega^2}-1;\, M_1=\frac{2\alpha_2A}{\Omega^2};\, M_2=\frac{3\alpha_3A^2}{2\Omega^2}\,. \tag{53}$$

Taking into account the expressions (46), (50) and (51), the auxiliary function α_1 is chosen in the form:

$$\alpha_1(\tau,C_1,C_2,...,C_6)=-C_1-2C_2\cos\tau-2C_3\cos 2\tau-2C_4\cos 3\tau, \tag{54}$$

but this form is not unique. Also, we can choose the auxiliary function α_1 in the following forms:

$$\alpha_1(\tau,C_1,C_2,C_3)=-C_1-2C_2\cos\tau-2C_3\cos 2\tau\,, \tag{55}$$

or

$$\alpha_1(\tau,C_1,C_2,C_3,C_4)=-C_1-2C_2\cos 3\tau-2C_3\cos 5\tau-2C_4\cos 7\tau \tag{56}$$

or yet

$$\begin{aligned}\alpha_1(\tau,C_1,C_2,...,C_7)=-C_1-2C_2\cos 2\tau-2C_3\cos 4\tau-\\ -2C_4\cos 6\tau-2C_5\cos 7\tau.\end{aligned} \tag{57}$$

Using only Eq. (54), Eq. (47) can be rewritten as:

$$\begin{aligned}&x_1''+x_1=P_0C_1+P_1C_2+P_2C_3-M_0+\\&+(P_1C_1+2P_0C_2+P_2C_2+P_1C_3+P_2C_4-M_1)\cos\tau+\\&+(P_2C_1+P_1C_2+2P_0C_3-M_2)\cos 2\tau+\\&+(P_2C_2+P_1C_3+2P_0C_4-M_3)\cos 3\tau+\\&+(P_2C_3+P_1C_4)\cos 4\tau+P_2C_4\cos 5\tau.\end{aligned} \tag{58}$$

Eliminating the secular term in Eq. (58), needs:

$$P_1C_1+(2P_0+P_2)C_2+P_1C_3+P_2C_4-M_1=0\,, \tag{59}$$

so that from Eqs. (52), (53) and (59), one gets:

$$\Omega^2=\alpha_1+\frac{2\alpha_2A(C_1+C_3)+\frac{3}{4}\alpha_3A^2(6C_2+4C_4-1)}{2(C_3+C_4)-1}\,. \tag{60}$$

From Eq. (38), with initial conditions:

$$x_1(0)=1, \quad x_1'(0)=0, \tag{61}$$

we obtain the first order approximate solution

$$\begin{aligned} x_1(\tau) &= \cos\tau + (P_0C_1 + P_1C_2 + P_2C_3)(1-\cos\tau) + \\ &+ \frac{1}{3}(P_2C_1 + P_1C_2 + 2P_0C_3 - M_2)(\cos\tau - \cos 2\tau) + \\ &+ \frac{1}{8}(P_2C_2 + P_1C_3 + 2P_0C_4 - M_3)(\cos\tau - \cos 3\tau) + \\ &+ \frac{1}{15}(P_2C_3 + P_1C_4)(\cos\tau - \cos 4\tau) + \frac{1}{24}P_2C_4(\cos\tau - \cos 5\tau). \end{aligned} \tag{62}$$

Having in view Eqs. (40) and (62), the approximate solution of Eqs. (29) and (33) is:

$$\begin{aligned} T(t) &= A\cos\Omega t + (P_0C_1 + P_1C_2 + P_2C_3)A(1-\cos\Omega t) + \\ &+ \frac{A}{3}(P_2C_1 + P_1C_2 + 2P_0C_3 - M_2)(\cos\Omega t - \cos 2\Omega t) + \\ &+ \frac{A}{8}(P_2C_2 + P_1C_3 + 2P_0C_4 - M_3)(\cos\Omega t - \cos 3\Omega t) + \\ &+ \frac{A}{15}(P_2C_3 + P_1C_4)(\cos\Omega t - \cos 4\Omega t) + \frac{A}{24}P_2C_4(\cos\Omega t - \cos 5\Omega t). \end{aligned} \tag{63}$$

The optimal convergence-control parameters $C_1 - C_4$ can be determined optimally using residual functional given by:

$$J(C_1, C_2, C_3, C_4) = \int_0^{\frac{2T}{\Omega}} \left[\ddot{T}(t) + \alpha_1 T(t) + \alpha_2 T^2(t) + \alpha_3 T^3(t)\right]^2 dt \tag{64}$$

and the values of parameters C_i can be obtained from conditions:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \frac{\partial J}{\partial C_3} = \frac{\partial J}{\partial C_4} = 0. \tag{65}$$

Follows that, the solution (63) in the first-approximation is well-determined.

5. NUMERICAL EXAMPLE

We will illustrate the aplicability, accuracy and effectiveness of the OPIM by comparing analytical approximate periodic solution with numerical integration results obtained using a fourth-order Runge-Kutta method.

For simply supported composite beam we obtain:

$$\alpha_1 = 97.4091, \quad \alpha_2 = 8.7331, \quad \alpha_3 = 24.4346\,. \tag{66}$$

Following the procedure described above, we obtain the optimal values of the unknown parameters for $A = 1$:

$$C_1 = 1.017029621067; C_2 = 0.01220592546; C_3 = 0.00748222975;$$
$$C_4 = -0.03676777504 \tag{67}$$

and $\Omega = 10.923352269336$.

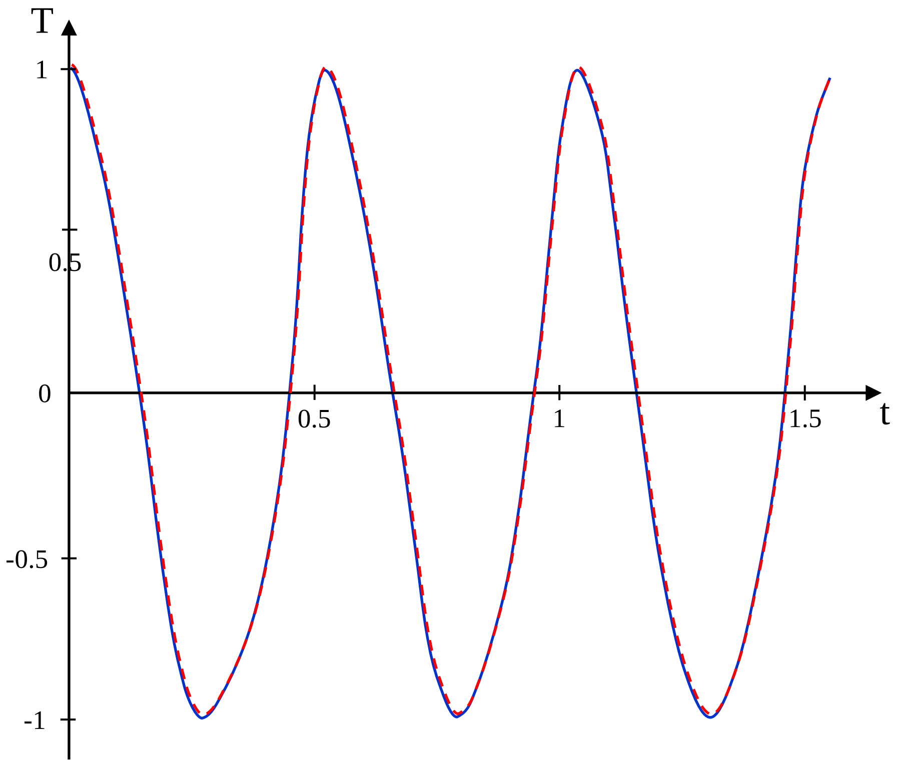

Fig. 1 – Comparison between the approximate solution (68) and numerical solution for $A = 1$ numerical, Eq. (68).

The first-order approximate solution in this case is:

$$\begin{aligned}T(t) &= 1.01595452114\cos\Omega t - 0.033383695 + 0.009070824099\cos 2\Omega t + \\ &+ 0.008123101509\cos 3\Omega t + 0.0001745884976\cos 4\Omega t + \\ &+ 0.0000596590222\cos 5\Omega t\end{aligned} \tag{68}$$

Comparison between the approximate solution (68) and numerical result is presented in Fig. 2.

It can be seen from Fig. 2 that the results obtained using OPIM are nearly identical with these obtained through numerical simulations.

6. CONCLUSIONS

In this paper, we proposed a reliable new procedure, namelly the Optimal Parametric Iteration Method (OPIM), which accelerate the rapid convergence of the approximate analytical solution of unsymmetrically laminated composite beams on Winkler-Pasternak foundation.

OPIM is based upon original construction of the solutions using a moderate number of the optimal convergence-control parameters which are components of the so-called auxiliary functions α_n, β_n and γ_n.

It is very impostant to mention that these optimal convergence-control parameters lead to a high precission comparing our approximate solutions with numerical results. The initial differential equation is reduced to only one linear differential equation, which depend on the initial approximation on the nonlinear operator ant its derivative calculated for initial approximation. The construction of equation which determine the first approximation is not unique.

We have a great freedom to choose the number of the optimal convergence-control parameters, the auxiliary functions and some terms from the nonlinear operator. The values of these parameters are determined optimally using rigorous mathematical procedure. OPIM is an iterative approach, which rapidly converges to exact solution after only one approximation.

Our technique lead to a very accurate results, is effective, explicit and simple. We remark the construction of the linear operator and of the auxiliary functions α_n, β_n and γ_n. These remarks are in fact the true power of our procedure to solving nonlinear whitout small or large parameters.

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