THE STUDY OF THE RELATIVE MOTION OF A MECHANISM USING LAGRANGE EQUATIONS

ROXANA ALEXANDRA PETRE , ION STROE

Abstract. The relative motion of a robotic arm formed by homogeneous bars of different lengths and masses, hinged to each other is studied. There are two cases studied: the first one when the first bar of the mechanism is hinged on a platform, which is considered initially to be fixed on the Earth's surface, for the second case, the platform is in rotation with respect to Earth. For all the analyzed cases the motion equations are determined using the Lagrangian formalism, applied in its traditional form, valid with respect to an inertial reference system, conventionally considered as fixed. However, in the second case, a generalized form of the formalism valid with respect to a non-inertial reference frame will also be applied. The numerical calculations were performed using a program developed using the MATLAB software.

Key words: Lagrange equations, relative motion, inertial or noninertial reference frame.

1. INTRODUCTION

In this paper the authors studied the relative motion of a robotic arm made up of hinged bars of different lengths and masses.

Two cases were considered: the first one, when the platform on which the first bar of the mechanism is hinged is fixed on the Earth's surface, and the second one, when the platform is in rotation with respect to Earth.

The first case corresponds to the motion of the mechanism with respect to a fixed reference frame, while the second case was analyzed with respect to an inertial, respectively with a non-inertial reference frame.

The motion equations were determined using the Lagrangian formalism [1–3], applied in its traditional form, valid with respect to an inertial reference frame. However, for the second case, a generalized form of the Lagrangian formalism will also be applied, valid with respect to a non-inertial reference frame [4–9].

Department of Mechanics, University "Politehnica" of Bucharest, Splaiul Independentei 313, Bucharest, 060042, Romania

The numerical simulations were performed using a program developed in the MATLAB software, where various sets of values for the system parameters were considered. For each set of values, the variation curves of the angles of rotation, angular velocities, and angular accelerations of the two bars, as well as the variation curves of the motor moments, for different values given to the angular velocity of the platform on which the first bar of the mechanism is situated were determined.

2. CONFIGURATION OF THE MECHANISM

A robotic arm consisting of two homogeneous bars is studied. The first bar, *OA*, is hinged in point *O* on the fixed element and the second bar, *AB*, which is articulated in point *A* on the first bar, rotates with respect to the *OA* bar about an axis unperpendicular to it situated in a horizontal plane, as shown in Fig. 1. The bars have the lengths l_1 and l_2 and the masses m_1 , m_2 , respectively.

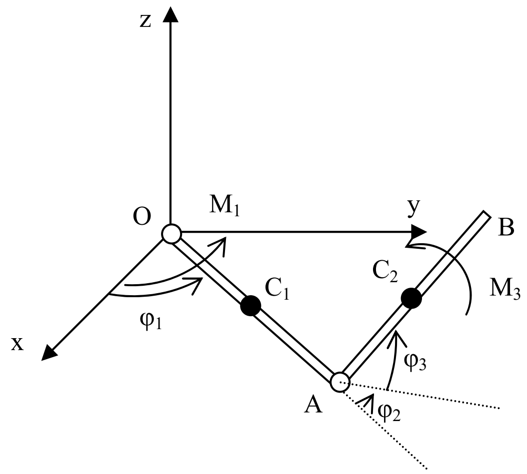

Fig. 1 – Mechanism with fixed platform.

The rotation plane of bar AB describes an angle

$$\varphi_2 = \text{const.} \tag{1}$$

with bar *OA*. If the angle $\varphi_2 = 0$, then the motion of the bar coincides with the bar studied in [8].

The bar *OA* is driven by the torque motor with the moment M_1, and the bar *AB* is driven by the bar *OA* via the torque motor with the moment M_3. The variations of the moments M_1 and M_3 are determined so that the motion of the robotic arm takes place according to the equations of motion:

$$\varphi_1(t) = A_1 \frac{\omega_0}{2\pi}\left(t - \frac{1}{\omega_0}\sin\left(\omega_0 t\right)\right), \tag{2}$$

$$\varphi_3(t) = A_3 \frac{\omega_0}{2\pi}\left(t - \frac{1}{\omega_0}\sin\left(\omega_0 t\right)\right). \tag{3}$$

2.1. MECHANISM HINGED ON A FIXED PLATFORM

Considering that the robotic arm is hinged on a fixed platform, the traditional form of the Lagrangian formalism [6 – 8] is applied:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E}{\partial \dot{\varphi}_1}\right) - \dfrac{\partial E}{\partial \varphi_1} = Q_1, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E}{\partial \dot{\varphi}_3}\right) - \dfrac{\partial E}{\partial \varphi_3} = Q_2, \end{cases} \tag{4}$$

where E is the kinetic energy of the system and Q_1 and Q_2 are the generalized forces of the system.

Considering the configuration of the mechanism the kinetic energy of the system, E, has the following form:

$$E = \frac{1}{2} J_O \dot{\varphi}_1^2 + \frac{1}{2} m_2 v_{C_2}^2 + \frac{1}{2} J_{C_2}(\dot{\varphi}_3^2 + \dot{\varphi}_1^2 \cos^2 \varphi_3), \tag{5}$$

where J_O represents the moment of inertia of the homogeneous bar OA with respect to point O, J_{C_2} is the moment of inertia of bar AB with respect to its center of mass, $\dot{\varphi}_1$ is the angular velocity of bar OA, $\dot{\varphi}_3$ is the angular velocity of bar AB and v_{C_2} is the linear velocity of the center of mass of bar AB with the following expression:

$$v_{C_2}^2 = \left(l_1^2 + \frac{l_2^2}{4}\cos^2\varphi_3 + l_1 l_2 \cos\varphi_2 \cos\varphi_3\right)\dot{\varphi}_1^2 + \frac{l_2^2}{4}\dot{\varphi}_3^2 - l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_3 \sin\varphi_2 \sin\varphi_3. \tag{6}$$

Thus, the kinetic energy of the system has the following form:

$$E=\frac{1}{2}\left(A_{11}+A_{22}\cos^2\varphi_3+2A_{12}\cos\varphi_2\cos\varphi_3\right)\dot{\varphi}_1^2+ \\ +\frac{1}{2}A_{22}\dot{\varphi}_3^2-A_{12}\dot{\varphi}_1\dot{\varphi}_3\sin\varphi_2\sin\varphi_3. , \tag{7}$$

where the following notations were used:

$$A_{11}=\left(\frac{m_1}{3}+m_2\right)l_1^2, \tag{8}$$

$$A_{22}=\frac{m_2}{3}l_2^2, \tag{9}$$

$$A_{12}=\frac{1}{2}m_2l_1l_2. \tag{10}$$

The generalized forces Q_1 and Q_2 from the right side of the Lagrange equations are determined using the virtual work:

$$\delta L=M_1\delta\varphi_1+M_2\delta\varphi_3, \tag{11}$$

thus, obtaining

$$\begin{cases}Q_1=M_1,\\ Q_2=M_3.\end{cases} \tag{12}$$

Replacing equations (5)-(12) in (4), the differential equations of motion were obtained:

$$\begin{cases}\left(A_{11}+A_{22}\cos^2\varphi_3+2A_{12}\cos\varphi_2\cos\varphi_3\right)\ddot{\varphi}_1-\\ -\left(2A_{22}\cos\varphi_3+2A_{12}\cos\varphi_2\right)\dot{\varphi}_1\dot{\varphi}_3\sin\varphi_3-\\ -A_{12}\ddot{\varphi}_3\sin\varphi_2\sin\varphi_3-A_{12}\dot{\varphi}_3^2\sin\varphi_2\cos\varphi_3=M_1,\\ A_{22}\ddot{\varphi}_3-A_{12}\ddot{\varphi}_1\sin\varphi_2\sin\varphi_3+\\ +\frac{1}{2}\left(2A_{22}\cos\varphi_3+2A_{12}\cos\varphi_2\right)\dot{\varphi}_1^2\sin\varphi_3=M_3.\end{cases} \tag{13}$$

2.2. MECHANISM HINGED ON A ROTATING PLATFORM

In this case, it is considered that the platform on which the mechanism is hinged is located on Earth, but in rotation with respect to it, with a constant angular velocity Ω, as shown in Fig. 2.

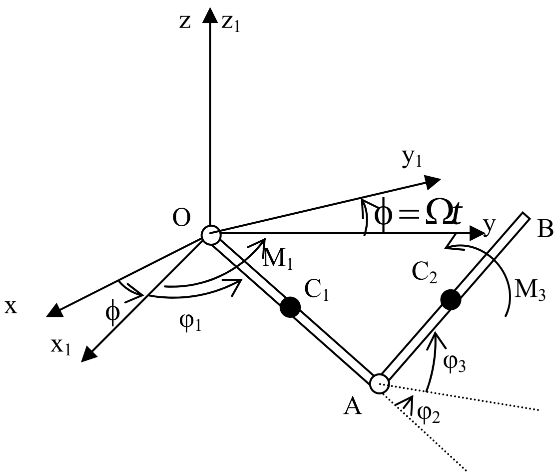

Fig. 2 – Mechanism on mobile platform.

For this case, the equations of motion were obtained for two subcases, that is, when the Lagrange equations with respect to a fixed reference frame *(O,x,y,z)* were used, and the second subcase when the Lagrange equations with respect to the mobile reference frame (O_1,x_1,y_1,z_1) were used.

2.2.1. The Lagrange equations with respect to a fixed reference frame

In this section the equations of motion of the robotic arm hinged on the mobile platform located on Earth, are determined using the Lagrange equations with respect to the fixed reference frame, (O,x,y,z). The platform has a rotation motion with respect to Earth with a constant angular velocity Ω.

The equations of motion are determined using (4), where the kinetic energy of the system has, in, this case, the form:

$$E=\frac{1}{2}J_O\left(\dot{\varphi}_1+\Omega\right)^2+\frac{1}{2}m_2v_{C_2}^2+\frac{1}{2}J_{C_2}\left[\dot{\varphi}_3^2+\left(\dot{\varphi}_1+\Omega\right)^2\cos^2\varphi_3\right], \quad (14)$$

where the linear velocity of the center of mass of bar *AB* has the following expression:

$$v_{c_2}^2 = \left(l_1^2 + \frac{l_2^2}{4}\cos^2\varphi_3 + l_1 l_2 \cos\varphi_2 \cos\varphi_3 \right)\left(\dot{\varphi}_1 + \Omega\right)^2 + \\ + \frac{l_2^2}{4}\dot{\varphi}_3^2 - l_1 l_2 \left(\dot{\varphi}_1 + \Omega\right)\dot{\varphi}_3 \sin\varphi_2 \sin\varphi_3. \tag{15}$$

Thus, by replacing all the terms in equation (14) the final form of the kinetic energy is obtained:

$$E = \frac{1}{2}\left(A_{11} + A_{22}\cos^2\varphi_3 + 2A_{12}\cos\varphi_2\cos\varphi_3\right)\left(\dot{\varphi}_1 + \Omega\right)^2 + \\ + \frac{1}{2}A_{22}\dot{\varphi}_3^2 - A_{12}\left(\dot{\varphi}_1 + \Omega\right)\dot{\varphi}_3 \sin\varphi_2 \sin\varphi_3. \tag{16}$$

The generalized forces Q_1 and Q_2 from the right side of the Lagrange equations are determined using the virtual work (11), as such they mentain their previous form (12).

By replacing equations (16) and (12) in system (4), the differential equations of motion are obtained:

$$\begin{cases} \left(A_{11} + A_{22}\cos^2\varphi_3 + 2A_{12}\cos\varphi_2\cos\varphi_3\right)\ddot{\varphi}_1 - \\ -2\left(A_{22}\cos\varphi_3 + A_{12}\cos\varphi_2\right)\left(\dot{\varphi}_1 + \Omega\right)\dot{\varphi}_3 \sin\varphi_3 - \\ -A_{12}\ddot{\varphi}_3 \sin\varphi_2 \sin\varphi_3 - A_{12}\dot{\varphi}_3^2 \sin\varphi_2 \cos\varphi_3 = M_1, \\ A_{22}\ddot{\varphi}_3 - A_{12}\ddot{\varphi}_1 \sin\varphi_2 \sin\varphi_3 + \\ +\left(A_{22}\cos\varphi_3 + A_{12}\cos\varphi_2\right)\left(\dot{\varphi}_1 + \Omega\right)^2 \sin\varphi_3 = M_3. \end{cases} \tag{17}$$

2.2.2. The Lagrange equations with respect to a mobile reference frame

In this subcase the equations of motion of the robotic arm which is hinged on the platform located on Earth and has a rotation motion with respect to it are determined, using the Lagrange equations with respect to a mobile reference frame, (O_1,x_1,y_1,z_1). This reference system has a constant angular velocity Ω. Since the motion is studied with respect to a mobile frame, a generalized form of the Lagrangian formalism valid in relation to a non-inertial reference frame [1, 4, 7–9] is used:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E_r}{\partial \dot{\varphi}_1}\right) - \dfrac{\partial E_r}{\partial \varphi_1} = Q_1 + Q_{1t} + Q_{1c}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E_r}{\partial \dot{\varphi}_3}\right) - \dfrac{\partial E_r}{\partial \varphi_3} = Q_3 + Q_{3t} + Q_{3c}, \end{cases} \tag{18}$$

where E_r is the kinetic energy of the system in relative motion, Q_1 and Q_3 are the generalized forces of form (12), Q_{1t} and Q_{3t} are the generalized transport forces,

$$Q_{kt} = -\sum_{i=1}^{n} m_i \overline{a_{ti}} \cdot \frac{\partial \overline{r_i}}{\partial q_k} \quad (k = 1,3), \tag{19}$$

and Q_{1c} and Q_{3c} represent the generalized Coriolis forces of the form

$$Q_{kc} = -\sum_{i=1}^{n} m_i \overline{a_{ci}} \cdot \frac{\partial \overline{r_i}}{\partial q_k} \quad (k = 1,3). \tag{20}$$

To obtain the generalized transport force, the kinetic energy for circular velocities is calculated:

$$E_C = E_C^{OA} + E_C^{AB} = \frac{1}{2} J_O \Omega^2 + \frac{1}{2} J_O^{AB} \Omega^2, \tag{21}$$

where J_O^{AB} is the moment of inertia of the bar *AB* with respect to point *O*.

For the robotic arm formed by the two homogeneous bars, the expressions found in [4],

$$Q_{kt} = Q_{kt}^{a} + Q_{kt}^{\varepsilon} + Q_{kt}^{\omega} \quad (k = 1,3), \tag{22}$$

and

$$Q_{kt}^{\omega} = \frac{\partial E_C}{\partial q_k} \quad (k = 1,3), \tag{23}$$

depend on the generalized coordinates of the system φ_1 and φ_3 :

$$\begin{cases} Q_{1t}^{\omega} = \dfrac{\partial E_C}{\partial \varphi_1} = 0, \\ Q_{2t}^{\omega} = \dfrac{\partial E_C}{\partial \varphi_2} = -\dfrac{1}{2}\Omega^2 \sin\varphi_3 \left(\dfrac{A_{22}}{2}\cos\varphi_3 \cos^2\varphi_2 + 2A_{12}\cos\varphi_2\right). \end{cases} \tag{24}$$

From Fig. 2 we conclude that the first bar is in rotation, thus the contribution of this element of the mechanism to the generalized Coriolis force is null, but for the second bar, which has an arbitrary relative motion, the generalized Coriolis force [4] has the following expression:

$$Q_{k_c}^{AB} = -2\overline{\omega_0}\left(m_2 \overline{v_{r_{c2}}} \times \frac{\partial \overline{v_{r_{c2}}}}{\partial \dot{q}_k} \right) - 2\overline{\omega_0}\,\overline{\overline{P_{C_2}}}\left(\overline{\omega_r} \times \frac{\partial \overline{\omega_r}}{\partial \dot{q}_k} \right), \tag{25}$$

where: $\overline{v_{r_{c2}}}$ is the relative velocity vector of the center of mass of bar *AB*, $\overline{\overline{P_{C_2}}}$ represents the tensor of the planar and centrifugal inertia moments for the second element of the mechanism.

This way, the Lagrange equations with respect to a non-inertial reference frame will be:

$$\begin{cases} \left(A_{11} + A_{22}\cos^2\varphi_3 + 2A_{12}\cos\varphi_2\cos\varphi_3\right)\ddot{\varphi}_1 - \\ -2\left(A_{22}\cos\varphi_3 + A_{12}\cos\varphi_2\right)\dot{\varphi}_1\dot{\varphi}_3\sin\varphi_3 - \\ -\sin\varphi_2\left(A_{12}\ddot{\varphi}_3\sin\varphi_3 + A_{12}\dot{\varphi}_3^2\cos\varphi_3\right) = M_1 + \dfrac{A_{22}}{4}\sin 2\varphi_3 + \Omega\dot{\varphi}_3 + \\ +\Omega\dot{\varphi}_3\dfrac{\sin 2\varphi_3}{2}\Big[\left(2A_{12}\cos\varphi_1 + 2A_{12}\cos\varphi_2\right) - \\ -\dfrac{3}{2}A_{22}\cos\varphi_3\left(\cos\left(\varphi_1+\varphi_2\right)-1\right)\Big], \\ \quad A_{22}\ddot{\varphi}_3 - \sin\varphi_3\left[A_{12}\ddot{\varphi}_1\sin\varphi_2 - \left(A_{22}\cos\varphi_3 + A_{12}\cos\varphi_2\right)\dot{\varphi}_1^2\right] = \\ \quad = M_3 - \dfrac{1}{2}\Omega^2\sin\varphi_3\cos\varphi_2\left(\dfrac{A_{22}}{2}\cos\varphi_2\cos\varphi_3 + 2A_{12}\right) - \\ \quad -\Omega\dot{\varphi}_1\sin\varphi_3\cos\varphi_3\left[\dfrac{A_{12}}{2}\left(\cos\varphi_2 + \cos\varphi_3\right) + \right. \\ \quad \left. + A_{22}\cos\varphi_3\left(\cos\left(\varphi_1+\varphi_3\right)+1\right)\right]. \end{cases} \tag{26}$$

3. NUMERICAL APPLICATIONS

The numerical calculations were performed using a program developed in the MATLAB software. For each set of numeric values considered for the system parameters: A_1, A_3, l_1, l_2, m_1, m_2 and φ_2, the variation curves of the angles of rotation, angular velocities and angular accelerations of the two bars, as well as the variation curves of the motor moments, for various values of the angular velocity Ω, were determined in Figs. 3–7.

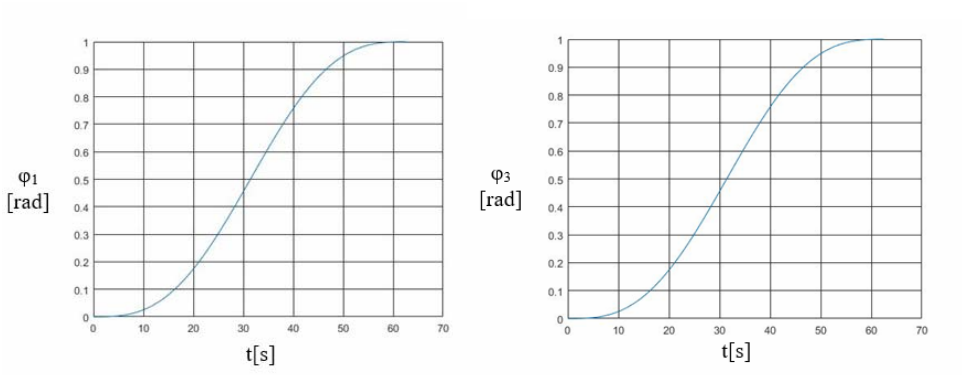

Fig. 3 – The law of motion for $A_1 = 1\,\text{rad}$, $A_3 = 1\,\text{rad}$.

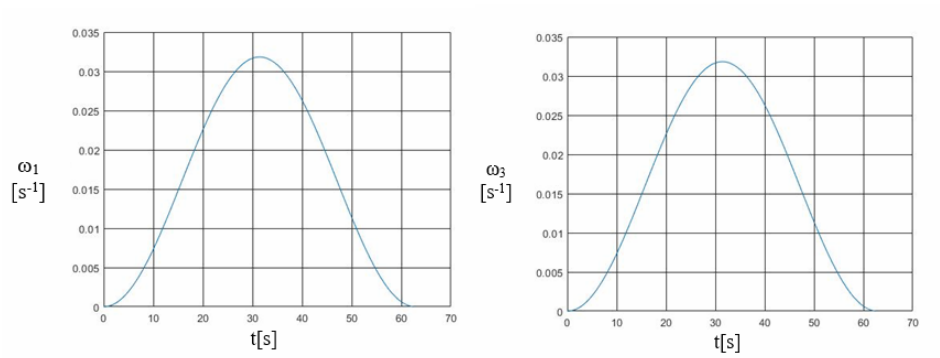

Fig. 4 – The angular velocities of the components of the mechanism when $A_1 = 1\,\text{rad}$, $A_3 = 1\,\text{rad}$.

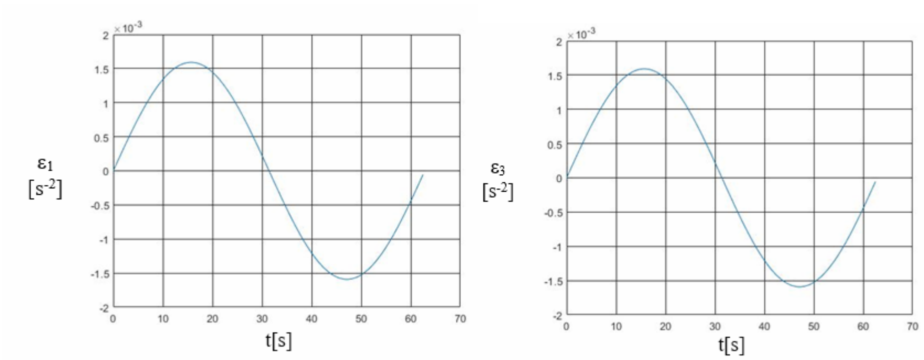

Fig. 5 – The angular accelerations of the components of the mechanism when $A_1 = 1\,\text{rad}$, $A_3 = 1\,\text{rad}$.

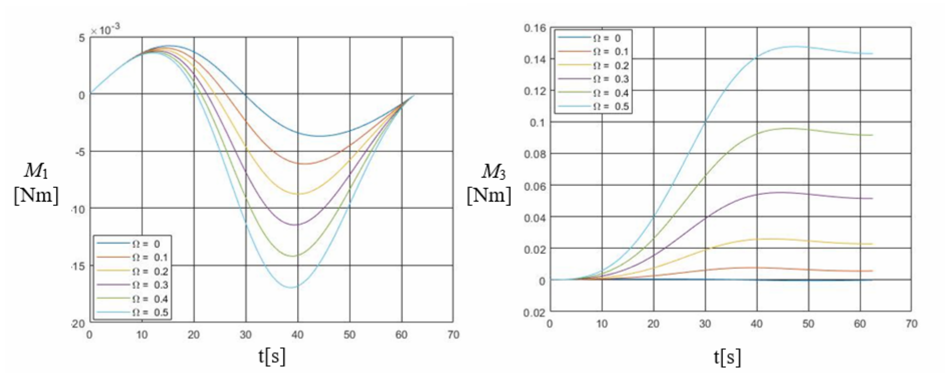

Fig. 6 – Motor moments for $A_1 = 1\,\text{rad}$, $A_3 = 1\,\text{rad}$, $l_1 = l_2 = 1\,\text{m}$, $m_1 = m_2 = 1\,\text{kg}$, $\varphi_2 = 0°$.

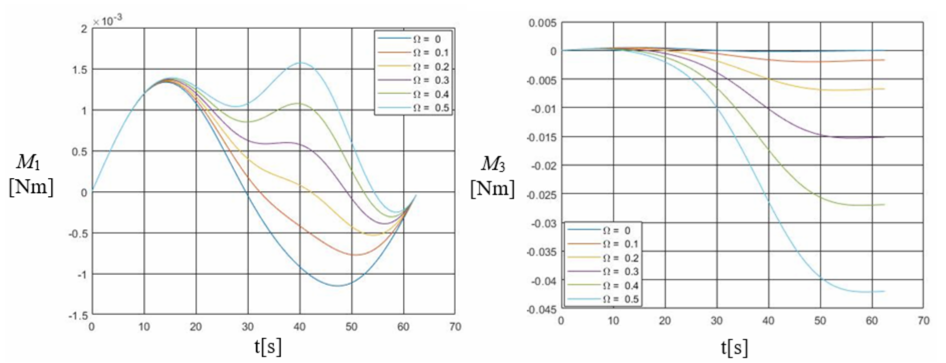

Fig. 7 – Motor moments for $A_1 = 1\,\text{rad}$, $A_3 = 1\,\text{rad}$, $l_1 = l_2 = 1\,\text{m}$, $m_1 = m_2 = 1\,\text{kg}$, $\varphi_2 = 15°$.

4. CONCLUSIONS

For the analyzed cases, the motion equations are determined using the Lagrangian formalism, applied in its traditional form, valid with respect to an inertial reference system, conventionally considered as fixed, while for the second case a generalized form of the formalism valid with respect to a non-inertial reference system has also been applied. It can be observed that the two versions of the Lagrangian formalism have led to the same results.

The numerical studies have shown that the values of the motor moments increase with the values of the amplitudes A_1 and A_3, and that the values of the motor moments generally, increase with the platform's angular velocity.

Received on December 13, 2021

REFERENCES

1. STROE, I., PÂRVU, P., *Trajectories in non-inertial reference farmes*, U.P.B. Sci. Bull. Series D, 72, *4*, ISSN - 1454-2358, 2010.
2. NGUYEN, T.-V., PETRE, R.-A., STROE, I., *Application of Lagrange Equations For Calculus of Internal Forces in a Mechanism*, U.P.B. Sci. Bull., Series D, 78, *4*, pp. 15–26, 2016.
3. NGUYEN, T.-V., PETRE R.-A., STROE I., *Calculus of axial force in a mechanism using Lagrange equations,* INCAS Bulletin, 8, *2*, pp. 97–108, 2016.
4. STROE, I., CRAIFALEANU, A., *Generalization of the Lagrange equations formalism, for motions with respect to non-inertial reference frames*, International Conference on Smart Systems in All Fields of the Life, Aerospace, Robotics, Mechanical Engineering, Manufacturing Systems, Biomechatronics, Neurorehabilitation and Human Motility, ICMERA 2014, Applied Mechanics and Materials, 656, pp. 171–180, 2014, Trans Tech Publications, Switzerland.
5. STROE, I., CRAIFALEANU, A., *Calculus of a compass robotic arm using Lagrange equations in non-inertial reference frames,* Proceedings of the International Conference of Aerospace Sciences (“AEROSPATIAL 2012”), Bucharest, 11–12 October, pp. 137–141, 2012.
6. STROE, I., STAICU, Ș., CRAIFALEANU, A., *Methods in dynamics of compass robotic arm*, Annual Symposium of the Institute of Solid Mechanics, SISOM, Bucharest, Romania, 2011.
7. PETRE, Roxana-Alexandra, STROE, Ion, CRAIFALEANU, Andrei, *Rotation-Rotation Mechanism moving with Respect to a Plane*, INCAS BULLETIN, 11, *4*, pp. 123–131, 2019.
8. PETRE, R. A., NICHIFOR, S. E., CRAIFALEANU, A., STROE, I., *Using Lagrange Equations to Study the Relative Motion of a Mechanism*, World Academy of Science, Engineering and Technology International Journal of Aerospace and Mechanical Engineering, 14, *10*, pp. 421–425, 2020.
9. STROE, Ion, STAICU, Stefan, *Calculus of Joint Forces Using Lagrange Equations and Principle of Virtual Work*, Proceedings of the Romanian Academy, Series A, 11, *3*, pp. 253–260, 2010.