ALMANSI-MICHELL PROBLEM FOR HETEROGENEOUS CHIRAL BEAMS

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Abstract. This paper is concerned with the deformation of a cylinder composed by two different chiral elastic materials welded together along the surface of separation. We study the equilibrium of a cylinder which is subjected to body loads, to tractions on the lateral surface and to resultant forces and moments on the ends. The Almansi-Michell problem, where the body loads and the surface loading are independent of the axial coordinate, is studied. The intended applications of the solution are to bone implants and to various compound cylinders. The chiral effects cannot be described within classical elasticity. In this paper we use the theory of isotropic chiral Cosserat elastic bodies. The three-dimensional problem is reduced to the study of plane problems. The solution is used to study the deformation of a uniformly loaded circular cylinder reinforced by a longitudinal rod. It is shown that a uniform pressure acting on the lateral surface of the cylinder produces a twist around its axis.

Key words: chiral materials, heterogeneous cylinders, uniformly loaded beams, flexure, reinforced circular cylinders.

1. INTRODUCTION

The mechanical behaviour of chiral materials is of fundamental interest to the fields of carbon nanotubes, auxetic materials and mechanics of bone. The chiral effects cannot be described within classical elasticity [1]. Throughout this paper we shall use the linear theory of isotropic chiral Cosserat elastic bodies. The Cosserat elastic solid was used as model for composite materials, carbon nanotubes and for bones (see, e.g., [2–9]). The study of mechanical behaviour of bones has received widespread attention [10]. Many investigations have been devoted to the deformation of bone implants (see, e.g., [11–13], and references therein).

In the theory of Cosserat elastic solids the Saint-Venant problem has been investigated in many papers (see, e.g., [14–16], and the references therein).

In the present paper we study the the equilibrium of a uniformly loaded cylinder composed by two different chiral elastic solids welded together along the surface of separation. The cylinder is subjected to body loads, to tractions on the lateral surface

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and to resultant forces and moments on the ends. When the body loads and the surface tractions on the lateral surface are independent of the axial coordinate, the problem is known as the Almansi-Michell problem. In the classical elasticity, the deformation of cylinders subjected to tractions on the lateral surface has been studied in various papers [17–19]. We reduce the three-dimensional problem to the study of plane problems. The paper is structured as follows. In Section 2 we present the basic equations of homogeneous and isotropic chiral bodies in the linear theory of Cosserat elastostatics and formulate the problem of heterogeneous cylinders. Section 3 is devoted to the generalized plane strain of the composed cylinder. In Section 4 we study the problem of loaded cylinders, when the body loads and the tractions on the lateral surface are independent of the axial coordinate. We show that the body loads and the tractions on the lateral surface produce extension, bending, torsion, flexure and a plane strain. In Section 5 the solution is used to study the deformation of a uniformly loaded circular cylinder reinforced by a longitudinal rod. It is shown that a uniform pressure acting on the lateral surface of the cylinder produces a twist around its axis.

2. BASIC EQUATIONS

The Cosserat theory studies continua in which each material point has the six degrees of freedom of a rigid body. The basic equations of the linear theory of isotropic chiral Cosserat elastic materials have been presented in [1,20]. Let us consider a body that in the undeformed state occupies the regular region B of euclidean three-dimensional space. We refer the body to a system of rectangular cartesian axes Ox_j. Latin subscripts (unless otherwise specified) are understood to range over the integers $(1,2,3)$. We call ∂B the boundary of B and designate the components of the outward unit normal of ∂B by n_k. We consider that the region B is occupied by a homogenenous and isotropic chiral Cosserat elastic material. Let u_k be the displacement vector, and let φ_k be the microrotation vector. The strain measures in the linear theory are given by [21]

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \ \ \kappa_{ij} = \varphi_{j,i}, \tag{1}$$

where ε_{ijk} is the alternating symbol, summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. We denote by t_{ij} the stress tensor, and by m_{ij} the couple stress tensor. The surface force and the surface moment acting at a regular point of ∂B are given by

$$t_i = t_{ji}n_j, \ \ m_i = m_{ji}n_j.$$

The homogenenous and isotropic chiral Cosserat elastic materials are characterized by the following constitutive equations

$$
\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu+\kappa)e_{ij} + \mu e_{ji} + C_1\kappa_{ss}\delta_{ij} + C_2\kappa_{ji} + C_3\kappa_{ij},\\
m_{ij} &= \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + C_1 e_{rr}\delta_{ij} + C_2 e_{ji} + C_3 e_{ij},
\end{aligned} \tag{2}
$$

where δ_{ij} is the Kronecker delta, and $\lambda,\mu,\kappa,\alpha,\beta,\gamma$ and C_j are constitutive constants. Let f_j be the body force, and g_j be the body couple. Then, the equations of equilibrium are given by

$$
t_{ji,j} + f_i = 0, \;\; m_{ji,j} + \varepsilon_{irs}t_{rs} + g_i = 0. \tag{3}
$$

We assume that the domain B from here on refers to the interior of a right cylinder of length h with the cross-section Σ and the lateral boundary Π. We suppose that Σ is a simply connected regular region. The cartesian coordinate frame is chosen in such a way that the x_3-axis is parallel to generator of the cylinder and x_1,x_2-plane contains one of the terminal cross sections. We call Σ_1 the cross-section located at $x_3 = 0$, and Σ_2 the cross-section that lies in the plane $x_3 = h$. We denote by L the boundary of the cross-section Σ_1. Let us suppose that the cylinder B is subjected to surface tractions on the lateral surface and to global conditions on the ends. Thus, on the boundary Π we have the conditions

$$
t_{\alpha i}n_\alpha = \widetilde{t}_i, \;\; m_{\alpha i}n_\alpha = \widetilde{m}_i \;\text{ on }\; \Pi, \tag{4}
$$

where Greek subscripts are understood to range over the integers $(1,2)$, and $\widetilde{t}_i$ and $\widetilde{m}_i$ are given functions. We assume that $\widetilde{t}_i$ and $\widetilde{m}_i$ are piecewise regular functions on Π. Let R_k and M_k be prescribed vectors representing the resultant force and the resultant moment about O of the tractions acting on Σ_1. Thus, for $x_3 = 0$ we have the following conditions

$$
\int_{\Sigma_1} t_{3\alpha}\,\mathrm{d}a = -R_\alpha, \tag{5}
$$

$$
\int_{\Sigma_1} t_{33}\,\mathrm{d}a = -R_3, \tag{6}
$$

$$
\int_{\Sigma_1} (x_\alpha t_{33} - \varepsilon_{3\alpha\beta}m_{3\beta})\,\mathrm{d}a = \varepsilon_{\alpha\beta 3}M_\beta, \tag{7}
$$

$$
\int_{\Sigma_1} (\varepsilon_{\alpha\beta 3}x_\alpha t_{3\beta} + m_{33})\,\mathrm{d}a = -M_3. \tag{8}
$$

Clearly, on the end Σ_2 there are tractions applied so as to satisfy the equilibrium conditions of the body. We denote by Γ a closed curve contained in Σ_1 which is the boundary of the regular domain Ω_2 contained in Σ_1. We assume that the curves Γ and L have no common points. Let Ω_1 be the domain bounded by L and Γ. We denote by B_ρ $(\rho = 1,2)$, the cylinder defined by $B_\rho = \{x : (x_1,x_2) \in \Omega_\rho,\, 0 < x_3 < h\}$. We assume that B_ρ is occupied by an isotropic chiral elastic material with the constitutive coefficients $\lambda^{(\rho)},\mu^{(\rho)},\kappa^{(\rho)},\alpha^{(\rho)},\beta^{(\rho)},\gamma^{(\rho)}$ and $C_j^{(\rho)}(\rho = 1,2)$. Let S be the surface of separation of the two materials. We suppose that the cylinder B is composed of two different materials which are welded together along S (Fig. 1). As in classical

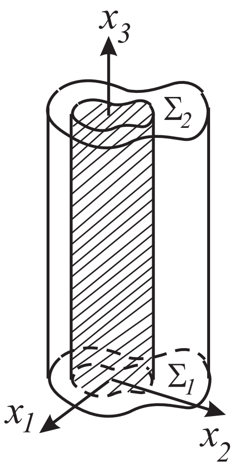

Fig. 1 – A heterogeneous cylinder.

elasticity [22] the displacement vector, the microrotation vector, the surface force and the surface moment must be continuous in passing from one medium to another. Thus we have the conditions

$$[u_j]_1 = [u_j]_2, [\varphi_j]_1 = [\varphi_j]_2, [t_{\alpha j}]_1 n_\alpha^0 = [t_{\alpha j}]_2 n_\alpha^0,\ [m_{\alpha j}]_1 n_\alpha^0 = [m_{\alpha j}]_2 n_\alpha^0, \tag{9}$$

on S. Here we have denoted by n_α^0 the components of the unit normal to Γ, outward to Ω_1, and we have indicated that the functions in brackets are calculated for the domains B_1 and B_2, respectively. In what follows we consider the problem of uniformly loaded cylinder which is characterized by the fact that the functions f_i, g_i, $\widetilde{t}_i$ and $\widetilde{m}_i$ are independent of the axial coordinate. We assume that

$$f_i = F_i^{(\rho)}(x_1,x_2), g_i = G_i^{(\rho)}(x_1,x_2),\ (x_1,x_2) \in \Omega_\rho,\ (\rho = 1,2). \tag{10}$$

The problem consists in finding the functions u_j and φ_j which satisfy the equations (1)–(3) on B_ρ $(\rho = 1,2)$, the conditions (4) on the lateral surface, the conditions (9) on the surface of separation, and the conditions (5)–(8) on the end Σ_1, when the body loads, the lateral loading and the constants R_k and M_k are given. Throughout this paper we assume that the elastic potential corresponding to the material that occupies the cylinder B_ρ $(\rho = 1,2)$ is a positive definite quadratic form in the strain tensors. The restrictions imposed by this assumption on the constitutive coefficients are presented in [1, 20].

3. PLANE STRAIN

In this section we investigate the two-dimensional deformation of a heterogenous cylinder. We say that the cylinder B is in a state of plane strain if the displacement vector and the microrotation vector are independent of the axial coordinate

$$u_j = u_j(x_1,x_2), \;\; \varphi_j = \varphi_j(x_1,x_2). \tag{11}$$

It follows from (1), (2) and (11) that in the case of a plane strain the stress tensor and the couple stress tensor are independent of the axial coordinate. The non-zero components of the strain tensors are given by

$$e_{\alpha i} = u_{i,\alpha} + \varepsilon_{i\alpha j}\varphi_j,\; e_{3i} = \varepsilon_{i3\beta}\varphi_\beta,\; \kappa_{\alpha i} = \varphi_{i,\alpha}. \tag{12}$$

The constitutive equations in the case of the plane strain are

$$\begin{aligned}
t_{\alpha\beta} &= \lambda^{(\rho)} e_{\eta\eta}\delta_{\alpha\beta} + (\mu^{(\rho)} + \kappa^{(\rho)})e_{\alpha\beta} + \mu^{(\rho)} e_{\beta\alpha} + \\
&\quad + C_1^{(\rho)}\kappa_{\nu\nu}\delta_{\alpha\beta} + C_2^{(\rho)}\kappa_{\beta\alpha} + C_3^{(\rho)}\kappa_{\alpha\beta},\\
t_{\alpha 3} &= (\mu^{(\rho)} + \kappa^{(\rho)})e_{\alpha 3} + \mu^{(\rho)} e_{3\alpha} + C_3^{(\rho)}\kappa_{\alpha 3},\\
t_{3\alpha} &= (\mu^{(\rho)} + \kappa^{(\rho)})e_{3\alpha} + \mu^{(\rho)} e_{\alpha 3} + C_2^{(\rho)}\kappa_{\alpha 3},\; t_{33} = \lambda^{(\rho)} e_{\eta\eta} + C_1^{(\rho)}\kappa_{\nu\nu},\\
m_{\nu\eta} &= \alpha^{(\rho)}\kappa_{\lambda\lambda}\delta_{\nu\eta} + \beta^{(\rho)}\kappa_{\eta\nu} + \gamma^{(\rho)}\kappa_{\nu\eta} + C_1^{(\rho)} e_{\lambda\lambda}\delta_{\nu\eta} + C_2^{(\rho)} e_{\eta\nu} + C_3^{(\rho)} e_{\nu\eta},\\
m_{\alpha 3} &= \gamma^{(\rho)}\kappa_{\alpha 3} + C_2^{(\rho)} e_{3\alpha} + C_3^{(\rho)} e_{\alpha 3},\\
m_{3\alpha} &= \beta^{(\rho)}\kappa_{\alpha 3} + C_2^{(\rho)} e_{\alpha 3} + C_3^{(\rho)} e_{3\alpha},\; m_{33} = \alpha^{(\rho)}\kappa_{\nu\nu} + C_1^{(\rho)} e_{\eta\eta} \text{ on } \Omega_\rho.
\end{aligned} \tag{13}$$

In the case of the plane strain the equations of equilibrium (3) take the form

$$t_{\alpha i,\alpha} + F_i^{(\rho)} = 0,\;\; m_{\alpha i,\alpha} + \varepsilon_{irs} t_{rs} + G_i^{(\rho)} = 0 \;\text{ on } \Omega_\rho. \tag{14}$$

The conditions (9) reduce to

$$\begin{aligned}
&[u_k]_1 = [u_k]_2,\; [\varphi_k]_1 = [\varphi_k]_2,\\
&[t_{\alpha i}]_1 n_\alpha^0 = [t_{\alpha i}]_2 n_\alpha^0,\; [m_{\alpha i}]_1 n_\alpha^0 = [m_{\alpha i}]_2 n_\alpha^0 \;\text{ on } \Gamma.
\end{aligned} \tag{15}$$

Since the functions t_{ij} and m_{ij} are independent of the axial coordinate, the conditions on the lateral surface become

$$[t_{\alpha i}]_1 n_\alpha = \widetilde{t}_i,\; [m_{\alpha i}]_1 n_\alpha = \widetilde{m}_i \;\text{ on } L. \tag{16}$$

The problem of plane strain consists in finding the displacement vector and the microrotation vector which satisfy the equations (12)-(14) on Ω_ρ, the conditions (15) on Γ, and the conditions (16) on L. We assume that $F_i^{(\rho)}, G_i^{(\rho)}$, $\widetilde{t}_i$ and $\widetilde{m}_i$ are prescribed

C^∞-fields. Following [22,23] the necessary and sufficient conditions for the existence of a solution to the boundary value problem (12)-(16) are

$$\int_L \widetilde{t}_i \mathrm{d}s + \sum_{\rho=1}^{2} \int_{\Omega_\rho} F_i^{(\rho)} \mathrm{d}a = 0, \tag{17}$$
$$\int_L (\varepsilon_{3\alpha\beta} x_\alpha \widetilde{t}_\beta + \widetilde{m}_3) \mathrm{d}s + \sum_{\rho=1}^{2} \int_{\Omega_\rho} (\varepsilon_{3\alpha\beta} x_\alpha F_\beta^{(\rho)} + G_3^{(\rho)}) \mathrm{d}a = 0.$$

Let p_j and q_j be prescribed functions on Γ. If we replace the conditions (15) by

$$\begin{gathered} [u_j]_1 = [u_j]_2, \quad [\varphi_j]_1 = [\varphi_j]_2, \\ [t_{\alpha i}]_1 n_\alpha^0 = [t_{\alpha i}]_2 n_\alpha^0 + p_i, \ [m_{\alpha i}]_1 n_\alpha^0 = [m_{\alpha i}]_2 n_\alpha^0 + q_i \text{ on } \Gamma, \end{gathered} \tag{18}$$

then the necessary and sufficient conditions for the existence of a solution are

$$\int_L \widetilde{t}_i \mathrm{d}s + \sum_{\rho=1}^{2} \int_{\Omega_\rho} F_i^{(\rho)} \mathrm{d}a + \int_\Gamma p_i \mathrm{d}s = 0, \tag{19}$$
$$\int_L (\varepsilon_{3\alpha\beta} x_\alpha \widetilde{t}_\beta + \widetilde{m}_3) \mathrm{d}s + \sum_{\rho=1}^{2} \int_{\Omega_\rho} (\varepsilon_{3\alpha\beta} x_\alpha F_\beta^{(\rho)} + G_3^{(\rho)}) \mathrm{d}a + \int_\Gamma (\varepsilon_{3\alpha\beta} x_\alpha p_\beta + q_3) \mathrm{d}s = 0.$$

If we know the functions u_k and φ_k, then from the constitutive equations (13) we can obtain the functions t_{3j} and m_{3j}. Then, we can determine the tractions over the ends Σ_1 and Σ_2 which maintain the cylinder in equilibrium. The equations of equilibrium can be expressed in the form

$$\begin{aligned} &(\mu^{(\rho)} + \kappa^{(\rho)}) \Delta u_\alpha + (\lambda^{(\rho)} + \mu^{(\rho)}) u_{\eta,\eta\alpha} + \kappa^{(\rho)} \varepsilon_{\alpha\beta 3} \varphi_{3,\beta} + C_3^{(\rho)} \Delta \varphi_\alpha + \\ &\qquad + (C_1^{(\rho)} + C_2^{(\rho)}) \varphi_{\eta,\eta\alpha} + F_\alpha^{(\rho)} = 0, \\ &(\mu^{(\rho)} + \kappa^{(\rho)}) \Delta u_3 + \kappa^{(\rho)} \varepsilon_{3\beta\alpha} \varphi_{\alpha,\beta} + C_3^{(\rho)} \Delta \varphi_3 + F_3^{(\rho)} = 0, \\ &C_3^{(\rho)} \Delta u_\nu + (C_1^{(\rho)} + C_2^{(\rho)}) u_{\beta,\beta\nu} + \kappa^{(\rho)} \varepsilon_{\nu\beta 3} u_{3,\beta} + \gamma^{(\rho)} \Delta \varphi_\nu + \\ &\qquad + (\alpha^{(\rho)} + \beta^{(\rho)}) \varphi_{\beta,\beta\nu} + 2(C_3^{(\rho)} - C_2^{(\rho)}) \varepsilon_{\nu\eta 3} \varphi_{3,\eta} - 2\kappa^{(\rho)} \varphi_\nu + G_\nu^{(\rho)} = 0, \\ &C_3^{(\rho)} \Delta u_3 + \kappa^{(\rho)} \varepsilon_{3\nu\eta} u_{\eta,\nu} + \gamma^{(\rho)} \Delta \varphi_3 + 2(C_3^{(\rho)} - C_2^{(\rho)}) \varepsilon_{3\nu\eta} \varphi_{\eta,\nu} - 2\kappa^{(\rho)} \varphi_3 + G_3^{(\rho)} = 0, \end{aligned} \tag{20}$$

on Ω_ρ $(\rho = 1,2)$. In the next section we will use four problems of plane strain, denoted by $A^{(k)}$ $(k = 1,2,3,4)$. We denote by $u_j^{(k)}, \varphi_j^{(k)}$, $e_{ij}^{(k)}, \kappa_{ij}^{(k)}, t_{ij}^{(k)}$ and $m_{ij}^{(k)}$ the displacement, microrotation strain tensors, stress tensor and couple stress tensor from the problem $A^{(k)}$ $(k = 1,2,3,4)$. The problem $A^{(1)}$ is characterized by the following external data

$$
\begin{aligned}
&F_i^{(\rho)} = \lambda^{(\rho)}\delta_{1i},\ G_i^{(\rho)} = (C_1^{(\rho)} + C_3^{(\rho)} - C_2^{(\rho)})\delta_{1i},\\
&\widetilde{t}_\alpha = -\lambda^{(1)}x_1 n_\alpha, \widetilde{t}_3 = C_2^{(1)}n_2,\ \widetilde{m}_\alpha = -C_1^{(1)}x_1 n_\alpha,\\
&\widetilde{m}_3 = \beta^{(1)}n_2, p_\alpha = (\lambda^{(2)} - \lambda^{(1)})x_1 n_\alpha^0, p_3 = (C_2^{(1)} - C_2^{(2)})n_2^0,\\
&q_\alpha = (C_1^{(2)} - C_1^{(1)})x_1 n_\alpha^0,\ q_3 = (\beta_1^{(2)} - \beta_1^{(1)})n_1^0.
\end{aligned}
\tag{21}
$$

In the problem $A^{(2)}$ the loads are given by

$$
\begin{aligned}
&F_i^{(\rho)} = \lambda^{(\rho)}\delta_{2i},\ G_i^{(\rho)} = (C_1^{(\rho)} + C_3^{(\rho)} - C_2^{(\rho)})\delta_{2i},\\
&\widetilde{t}_\alpha = -\lambda^{(1)}x_2 n_\alpha,\ \widetilde{t}_3 = -C_2^{(1)}n_1,\ \widetilde{m}_\alpha = -C_1^{(1)}x_2 n_\alpha,\\
&\widetilde{m}_3 = -\beta^{(1)}n_1,\ p_\alpha = (\lambda^{(2)} - \lambda^{(1)})x_2 n_\alpha^0,\ p_3 = (C_2^{(2)} - C_2^{(1)})n_1^0,\\
&q_\alpha = (C_1^{(2)} - C_1^{(1)})x_2 n_\alpha^0,\ q_3 = (\beta^{(2)} - \beta^{(1)})n_1^0.
\end{aligned}
\tag{22}
$$

The solution of the problem $A^{(3)}$ corresponds to the following loading

$$
\begin{aligned}
&F_i^{(\rho)} = 0,\ G_i^{(\rho)} = 0, \widetilde{t}_\alpha = -\lambda^{(1)}n_\alpha, \widetilde{t}_3 = 0,\\
&\widetilde{m}_\alpha = -C_1^{(1)}n_\alpha, \widetilde{m}_3 = 0,\ p_\alpha = (\lambda^{(2)} - \lambda^{(1)})n_\alpha^0,\\
&p_3 = 0,\ q_\alpha = (C_1^{(2)} - C_1^{(1)})n_\alpha^0,\ q_3 = 0.
\end{aligned}
\tag{23}
$$

In the problem $A^{(4)}$ the external data are

$$
\begin{aligned}
&F_i^{(\rho)} = 0,\ G_\alpha^{(\rho)} = -\kappa^{(\rho)}x_\alpha,\ G_3^{(\rho)} = 0,\\
&\widetilde{t}_\alpha = -C_1^{(1)}n_\alpha,\ \widetilde{t}_3 = -\mu^{(1)}\varepsilon_{3\eta\alpha}x_\eta n_\alpha,\ \widetilde{m}_\beta = -\alpha^{(1)}n_\beta,\\
&\widetilde{m}_3 = C_2^{(1)}\varepsilon_{3\nu\alpha}x_\alpha n_\nu,\ p_\alpha = (C_1^{(2)} - C_1^{(1)})n_\alpha^0,\\
&p_3 = (\mu^{(2)} - \mu^{(1)})\varepsilon_{3\nu\alpha}x_\nu n_\alpha^0,\ q_\nu = (\alpha^{(2)} - \alpha^{(1)})n_\nu^0,\\
&q_3 = (C_2^{(2)} - C_2^{(1)})\varepsilon_{3\alpha\beta}x_\alpha n_\beta^0.
\end{aligned}
\tag{24}
$$

Let us note that the necessary and sufficient conditions for the existence of the solution are satisfied for each problem $A^{(k)}$ $(k = 1,2,3,4)$. The problems $A^{(k)}$ have been introduced with the occasion of solving of torsion problem [14].

4. ALMANSI-MICHELL PROBLEM

In this section we establish the solution of the Almansi-Michell problem for cylinders composed by different materials. Following the results established in the case of achiral materials [18], we seek the solution of the problem formulated in Section 2 in the form

$$\begin{aligned}
u_\alpha &= -\frac{1}{2}a_\alpha x_3^2 - \frac{1}{6}b_\alpha x_3^3 - \frac{1}{24}c_\alpha x_3^4 + \varepsilon_{3\beta\alpha}(a_4 x_3 + \frac{1}{2}b_4 x_3^2 + \frac{1}{6}c_4 x_3^3)x_\beta + \\
&+ \sum_{k=1}^{4}(a_k + b_k x_3 + \frac{1}{2}c_k x_3^2)u_\alpha^{(k)} + w_\alpha(x_1,x_2) + x_3 v_\alpha(x_1,x_2),\\
u_3 &= (a_1 x_1 + a_2 x_2 + a_3)x_3 + \frac{1}{2}(b_1 x_1 + b_2 x_2 + b_3)x_3^2 + \\
&+ \frac{1}{6}(c_1 x_1 + c_2 x_2 + c_2)x_3^3 + \sum_{k=1}^{4}(a_k + b_k x_3 + \\
&+ \frac{1}{2}c_k x_3^2)u_3^{(k)} + w_3(x_1,x_2) + x_3 v_3(x_1,x_2),\\
\varphi_\alpha &= \varepsilon_{3\alpha\beta}(a_\beta x_3 + \frac{1}{2}b_\beta x_3^2 + \frac{1}{6}c_\beta x_3^3) + \sum_{k=1}^{4}(a_k + b_k x_3 + \\
&+ \frac{1}{2}c_k x_3^2)\varphi_\alpha^{(k)} + \chi_\alpha(x_1,x_2) + x_3\psi_\alpha(x_1,x_2),\\
\varphi_3 &= a_4 x_3 + \frac{1}{2}b_4 x_3^2 + \frac{1}{6}c_4 x_3^3 + \sum_{k=1}^{4}(a_k + b_k x_3 + \frac{1}{2}c_k x_3^2)\varphi_3^{(k)} + \\
&+ \chi_3(x_1,x_2) + x_3\psi_3(x_1,x_2),
\end{aligned} \tag{25}$$

where $u_j^{(k)}$ and $\varphi_j^{(k)}$ are the displacements and the microrotations from the problem $A^{(k)}$ $(k=1,2,3,4)$, introduced in Section 3, v_j, ψ_j, w_j and χ_j are unknown functions which are independent of the axial coordinate, and a_k, b_k and c_k $(k=1,2,3,4)$, are unknown constants.

Let us consider a state of plane strain of the cylinder B characterized by the displacements w_j and the microrotations χ_j. The strain tensors in this problem are given by

$$\xi_{\alpha i} = w_{i,\alpha} + \varepsilon_{i\alpha j}\chi_j, \quad \xi_{3j} = \varepsilon_{3\beta j}\chi_\beta, \quad \eta_{\alpha i} = \chi_{i,\alpha}. \tag{26}$$

We denote by τ_{ij} and μ_{ij} the stress tensor and the couple stress tensor corresponding to the functions w_j and χ_j,

$$\begin{aligned}
\tau_{\alpha\beta} &= \lambda^{(\rho)}\xi_{\eta\eta}\delta_{\alpha\beta} + (\mu^{(\rho)}+\kappa^{(\rho)})\xi_{\alpha\beta} + \mu^{(\rho)}\xi_{\beta\alpha} + C_1^{(\rho)}\eta_{\nu\nu}\delta_{\alpha\beta} + \\
&\quad + C_2^{(\rho)}\eta_{\beta\alpha} + C_3^{(\rho)}\eta_{\alpha\beta},\\
\tau_{\alpha 3} &= (\mu^{(\rho)}+\kappa^{(\rho)})\xi_{\alpha 3} + \mu^{(\rho)}\xi_{3\alpha} + C_3^{(\rho)}\eta_{\alpha 3},\\
\tau_{3\alpha} &= (\mu^{(\rho)}+\kappa^{(\rho)})\xi_{3\alpha} + \mu^{(\rho)}\xi_{\alpha 3} + C_2^{(\rho)}\eta_{\alpha 3},\\
\tau_{33} &= \lambda^{(\rho)}\xi_{\eta\eta} + C_1^{(\rho)}\eta_{\nu\nu},\ \mu_{33} = \alpha^{(\rho)}\eta_{\nu\nu} + C_1^{(\rho)}\xi_{\nu\nu},\\
\mu_{\nu\lambda} &= \alpha^{(\rho)}\eta_{\kappa\kappa}\delta_{\nu\lambda} + \beta^{(\rho)}\eta_{\lambda\nu} + \gamma^{(\rho)}\eta_{\nu\lambda} + C_1^{(\rho)}\xi_{\kappa\kappa}\delta_{\nu\lambda} + C_2^{(\rho)}\xi_{\lambda\nu} + C_3^{(\rho)}\xi_{\nu\lambda},\\
\mu_{\alpha 3} &= \gamma^{(\rho)}\eta_{\alpha 3} + C_2^{(\rho)}\xi_{3\alpha} + C_3^{(\rho)}\xi_{\alpha 3},\\
\mu_{3\alpha} &= \beta^{(\rho)}\eta_{\alpha 3} + C_2^{(\rho)}\xi_{\alpha 3} + C_3^{(\rho)}\xi_{3\alpha},\ \text{on } \Omega_\rho.
\end{aligned} \tag{27}$$

Similarly, we consider a plane strain of the cylinder B in which the displacement vector is v_k and the microrotation vector is ψ_k. We introduce the notations

$$\gamma_{\alpha i} = v_{i,\alpha} + \varepsilon_{i\alpha j}\psi_j,\ \gamma_{3i} = \varepsilon_{i3\beta}\psi_\beta,\ \zeta_{\alpha j} = \psi_{j,\alpha}. \tag{28}$$

The stress tensor σ_{ij} and the couple stress tensor ν_{ij} associated to the strain measures γ_{jk} and $\zeta_{\alpha i}$ are given by

$$\begin{aligned}
\sigma_{\alpha\beta} &= \lambda^{(\rho)}\gamma_{\kappa\kappa}\delta_{\alpha\beta} + (\mu^{(\rho)}+\kappa^{(\rho)})\gamma_{\alpha\beta} + \mu^{(\rho)}\gamma_{\beta\alpha} + C_1^{(\rho)}\zeta_{\kappa\kappa}\delta_{\alpha\beta} + \\
&\quad + C_2^{(\rho)}\zeta_{\beta\alpha} + C_3^{(\rho)}\zeta_{\alpha\beta},\\
\sigma_{\alpha 3} &= (\mu^{(\rho)}+\kappa^{(\rho)})\gamma_{\alpha 3} + \mu^{(\rho)}\gamma_{3\alpha} + C_3^{(\rho)}\zeta_{\alpha 3}, \sigma_{3\alpha} = (\mu^{(\rho)}+\kappa^{(\rho)})\gamma_{3\alpha} + \\
&\quad + \mu^{(\rho)}\gamma_{\alpha 3} + C_2^{(\rho)}\zeta_{\alpha 3},\\
\sigma_{33} &= \lambda^{(\rho)}\gamma_{\kappa\kappa} + C_1^{(\rho)}\zeta_{\nu\nu},\ \nu_{33} = \alpha^{(\rho)}\zeta_{\kappa\kappa} + C_1^{(\rho)}\gamma_{\kappa\kappa},\\
\nu_{\eta\lambda} &= \alpha^{(\rho)}\zeta_{\kappa\kappa}\delta_{\eta\lambda} + \beta^{(\rho)}\zeta_{\lambda\nu} + \gamma^{(\rho)}\zeta_{\nu\lambda} + C_1^{(\rho)}\gamma_{\kappa\kappa}\delta_{\eta\lambda} + \\
&\quad + C_2^{(\rho)}\gamma_{\lambda\nu} + C_3^{(\rho)}\gamma_{\eta\lambda},\\
\nu_{\alpha 3} &= \gamma^{(\rho)}\zeta_{\alpha 3} + C_2^{(\rho)}\gamma_{3\alpha} + C_3^{(\rho)}\gamma_{\alpha 3}, \nu_{3\alpha} = \beta^{(\rho)}\zeta_{\alpha 3} + C_2^{(\rho)}\gamma_{\alpha 3} + \\
&\quad + C_3^{(\rho)}\gamma_{3\alpha},\ \text{on } \Omega_\rho.
\end{aligned} \tag{29}$$

With the help of the equations (1), (25)–(29), we find from (2) that the stress tensor

t_{ij} is given by

$$
\begin{aligned}
t_{\alpha\beta} = {} & \lambda^{(\rho)}[a_1x_1 + a_2x_2 + a_3 + (b_1x_1 + b_2x_2 + b_3)x_3 + \\
& + \frac{1}{2}(c_1x_1 + c_2x_2 + c_3)x_3^2]\delta_{\alpha\beta} + C_1^{(\rho)}\delta_{\alpha\beta}(a_4 + b_4x_3 + \frac{1}{2}c_4x_3^2) + \\
& + \sum_{k=1}^{4}(a_k + b_kx_3 + \frac{1}{2}c_kx_3^2)t_{\alpha\beta}^{(k)} + \tau_{\alpha\beta} + x_3\sigma_{\alpha\beta} + \\
& + [\lambda^{(\rho)}v_3 + C_1^{(\rho)}\psi_3 + \sum_{k=1}^{4}(b_k + c_kx_3)(\lambda^{(\rho)}u_3^{(k)} + C_1^{(\rho)}\varphi_3^{(\rho)})]\delta_{\alpha\beta}, \\
t_{\alpha 3} = {} & \mu^{(\rho)}(a_4 + b_4x_3 + \frac{1}{2}c_4x_3^2)\varepsilon_{3\beta\alpha}x_\beta + C_2^{(\rho)}(a_\beta + b_\beta x_3 + \frac{1}{2}c_\beta x_3^2)\varepsilon_{3\alpha\beta} + \\
& + \sum_{k=1}^{4}(a_k + b_kx_3 + \frac{1}{2}c_kx_3^2)t_{\alpha 3}^{(k)} + \tau_{\alpha 3} + x_3\sigma_{\alpha 3} + \mu^{(\rho)}v_\alpha + C_2^{(\rho)}\psi_\alpha + \\
& + \sum_{k=1}^{4}(b_k + c_kx_3)(\mu^{(\rho)}v_\alpha^{(k)} + C_2^{(\rho)}\varphi_\alpha^{(k)}), \qquad (30) \\
t_{3\alpha} = {} & (\mu^{(\rho)} + \kappa^{(\rho)})(a_4 + b_4x_3 + \frac{1}{2}c_4x_3^2)\varepsilon_{3\beta\alpha}x_\beta + \\
& + C_3^{(\rho)}\varepsilon_{3\alpha\beta}(a_\beta + b_\beta x_3 + \frac{1}{2}c_\beta x_3^2) + \\
& + \sum_{k=1}^{4}(a_k + b_kx_3 + \frac{1}{2}c_kx_3^2)t_{3\alpha}^{(k)} + \tau_{3\alpha} + x_3\sigma_{3\alpha} + (\mu^{(\rho)} + \kappa^{(\rho)})v_\alpha + C_3^{(\rho)}\psi_\alpha + \\
& + \sum_{k=1}^{4}(b_k + c_kx_3)[(\mu^{(\rho)} + \kappa^{(\rho)})u_\alpha^{(k)} + C_3\varphi_\alpha^{(k)}], \\
t_{33} = {} & (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})[a_1x_1 + a_2x_2 + a_3 + (b_1x_1 + b_2x_2 + b_3)x_3 + \\
& + \frac{1}{2}(c_1x_1 + c_2x_2 + c_3)x_3^2] + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})(a_4 + b_4x_3 + \\
& + \frac{1}{2}c_4x_3^2) + \sum_{k=1}^{4}(a_k + b_kx_3 + \frac{1}{2}c_kx_3^2)t_{33}^{(k)} + \tau_{33} + x_3\sigma_{33} + \\
& + (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})v_3 + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\psi_3 + \\
& + \sum_{k=1}^{4}(b_k + c_kx_3)[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})u_3^{(k)} + \\
& + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})]\varphi_3^{(k)}, \text{ on } \Omega_\rho,
\end{aligned}
$$

where $t_{ij}^{(k)}$ is the stress tensor from the problem $A^{(k)}$ $(k = 1,2,3,4)$. In view of the constitutive equations (2), we find that the couple stress tensor has the form

$$
\begin{aligned}
m_{\eta\lambda} &= \alpha^{(\rho)}(a_4+b_4x_3+\frac{1}{2}c_4x_3^2)\delta_{\eta\lambda}+C_1^{(\rho)}[a_1x_1+a_2x_2+a_3+\\
&+(b_1x_1+b_2x_2+b_3)x_3+\frac{1}{2}(c_1x_1+c_2x_2+c_3)x_3^2]\delta_{\eta\lambda}+\sum_{k=1}^{4}(a_k+b_kx_3+\\
&+\frac{1}{2}c_kx_3^2)m_{\eta\lambda}^{(k)}+\mu_{\eta\lambda}+x_3\nu_{\eta\lambda}+(\alpha^{(\rho)}\psi_3+C_1^{(\rho)}v_3)\delta_{\eta\lambda}+\\
&+\delta_{\eta\lambda}\sum_{k=1}^{4}(b_k+c_kx_3)(\alpha^{(\rho)}\varphi_3^{(k)}+C_1^{(\rho)}u_3^{(k)}),\\
m_{\eta 3} &= \beta\varepsilon_{3\eta\lambda}(a_\lambda+b_\lambda x_3+\frac{1}{2}c_\lambda x_3^2)+C_2^{(\rho)}\varepsilon_{3\lambda\eta}x_\lambda(a_4+b_4x_3+\\
&+\frac{1}{2}c_4x_3^2)+\sum_{k=1}^{4}(a_k+b_kx_3+\frac{1}{2}c_kx_3^2)m_{\eta 3}^{(k)}+\mu_{\eta 3}+x_3\nu_{\eta 3}+\\
&+\beta^{(\rho)}\psi_\eta+C_2^{(\rho)}v_\eta+\sum_{k=1}^{4}(b_k+c_kx_3)(\beta^{(\rho)}\psi_\eta^{(k)}+C_2^{(\rho)}u_\eta^{(k)}),\\
m_{3\eta} &= \gamma^{(\rho)}\varepsilon_{3\eta\lambda}(a_\lambda+b_\lambda x_3+\frac{1}{2}c_\lambda x_3^2)+C_3^{(\rho)}\varepsilon_{3\lambda\eta}x_\lambda(a_4+\\
&+b_4x_3+\frac{1}{2}c_4x_3^2)+\sum_{k=1}^{4}(a_k+b_kx_3+\frac{1}{2}c_kx_3^2)m_{3\eta}^{(k)}+\\
&+\mu_{3\eta}+x_3\nu_{3\eta}+\gamma^{(\rho)}\psi_\eta+C_3^{(\rho)}v_\eta+\\
&+\sum_{k=1}^{4}(b_k+c_kx_3)(\gamma^{(\rho)}\psi_\eta^{(k)}+C_3^{(\rho)}u_\eta^{(k)}),\\
m_{33} &= (C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})[a_1x_1+a_2x_2+a_3+(b_1x_1+b_2x_2+b_3)x_3+\\
&+\frac{1}{2}(c_1x_1+c_2x_2+c_3)x_3^2]+(\alpha^{(\rho)}+\beta^{(\rho)}+\gamma^{(\rho)})(a_4+b_4x_3+\\
&+\frac{1}{2}c_4x_3^2)+\sum_{k=1}^{4}(a_k+b_kx_3+\frac{1}{2}c_kx_3^2)m_{33}^{(k)}+\mu_{33}+\\
&+x_3\nu_{33}+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})v_3+(\alpha+\beta+\gamma)\psi_3+\\
&+\sum_{k=1}^{4}(b_k+c_kx_3)[(\alpha^{(\rho)}+\beta^{(\rho)}+\gamma^{(\rho)})\varphi_3^{(k)}+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})u_3^{(k)}],\ \text{on } \Omega_\rho.
\end{aligned}
\tag{31}
$$

Let us substitute the functions t_{ij} and m_{ij} into the equilibrium equations (3). If we use the equilibrium equations from the problems $A^{(k)}$ and the external data defined in (21)–(24), then the equations (3) reduce to

$$\tau_{\beta i,\beta}+f_i^{(\rho)}=0,\ \mu_{\beta i,\beta}+\varepsilon_{irs}\tau_{rs}+g_i^{(\rho)}=0 \ \text{ on } \Omega_\rho, \tag{32}$$

and

$$\sigma_{\beta i,\beta}+h_i^{(\rho)}=0,\ \ \nu_{\beta i,\beta}+\varepsilon_{irs}\sigma_{rs}+l_i^{(\rho)}=0\ \text{ on } \Omega_\rho. \tag{33}$$

In the equations (32) and (33) we have used the notations

$$\begin{aligned}
f_i^{(\rho)}&=F_i^{(\rho)}+(\mu^{(\rho)}+\kappa^{(\rho)})\varepsilon_{3\eta\alpha}b_4x_\eta+C_3^{(\rho)}\varepsilon_{3\alpha\eta}b_\eta+\\
&+\lambda^{(\rho)}v_{3,\alpha}+C_1^{(\rho)}\psi_{3,\alpha}+\sigma_{3\alpha}+\sum_{k=1}^{4}\{(t_{3\alpha}^{(k)}+\lambda^{(\rho)}u_{3,\alpha}^{(k)}+\\
&+C_1^{(\rho)}\varphi_{3,\alpha}^{(k)})b_k+[(\mu^{(\rho)}+\kappa^{(\rho)})u_\alpha^{(k)}+C_3^{(\rho)}\varphi_\alpha^{(k)}]c_k\},\\
f_3^{(\rho)}&=F_3^{(\rho)}+(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})(b_1x_1+b_2x_2+b_3)+(C_1^{(\rho)}+C_2^{(\rho)}+\\
&+C_3^{(\rho)})b_4+\mu^{(\rho)}v_{\alpha,\alpha}+C_2^{(\rho)}\psi_{\alpha,\alpha}+\sigma_{33}+\sum_{k=1}^{4}\{(t_{33}^{(k)}+\mu^{(\rho)}u_{\alpha,\alpha}^{(k)}+\\
&+C_2^{(\rho)}\varphi_{\alpha,\alpha}^{(k)})b_k+[(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})u_3^{(k)}+\\
&+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})\varphi_3^{(k)}]c_k\},\\
g_\lambda^{(\rho)}&=G_\lambda^{(\rho)}+\alpha^{(\rho)}\psi_{3,\alpha}+C_1^{(\rho)}v_{3,\lambda}+\nu_{3\lambda}+\varepsilon_{3\lambda\nu}[\gamma^{(\rho)}b_\eta-\\
&-C_3^{(\rho)}b_4x_\eta-\kappa^{(\rho)}v_\eta-(C_3^{(\rho)}-C_2^{(\rho)})\psi_\eta]+\\
&+\sum_{k=1}^{4}\{[m_{3\lambda}^{(k)}+\alpha^{(\rho)}\varphi_{3,\lambda}^{(k)}+C_1^{(\rho)}u_{3,\lambda}^{(k)}+\varepsilon_{3\lambda\eta}(C_2^{(\rho)}-C_3^{(\rho)})\varphi_\lambda^{(k)}-\\
&-\kappa^{(\rho)}\varepsilon_{3\lambda\eta}u_\eta^{(k)}]b_k+(\gamma^{(\rho)}\psi_\lambda^{(k)}+C_3^{(\rho)}u_\lambda^{(k)})c_k\},\\
g_3^{(\rho)}&=G_3^{(\rho)}+\beta^{(\rho)}\psi_{\alpha,\alpha}+C_2^{(\rho)}\nu_{\alpha,\alpha}+\nu_{33}+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})(b_1x_1+\\
&+b_2x_2+b_3)+(\alpha^{(\rho)}+\beta^{(\rho)}+\gamma^{(\rho)})b_4+\sum_{k=1}^{4}\{[m_{33}^{(k)}+\\
&+\beta^{(\rho)}\psi_{\alpha,\alpha}^{(k)}+C_2^{(\rho)}u_{\alpha,\alpha}^{(k)}]b_k+[(\alpha^{(\rho)}+\beta^{(\rho)}+\gamma^{(\rho)})\varphi_3^{(k)}+\\
&+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})u_3^{(k)}]c_k\},\ \text{ on }\ \Omega_\rho,
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
h_\alpha^{(\rho)}&=(\mu^{(\rho)}+\kappa^{(\rho)})c_4\varepsilon_{3\eta\alpha}x_\eta+C_3^{(\rho)}\varepsilon_{3\alpha\eta}c_\eta+\sum_{k=1}^{4}(t_{3\alpha}^{(k)}+\lambda^{(\rho)}u_{3,\alpha}^{(k)}+\\
&+C_1^{(\rho)}\varphi_{3,\alpha}^{(k)})c_k,\\
h_3^{(\rho)}&=(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})(c_1x_1+c_2x_2+c_3)+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})c_4+\\
&+\sum_{k=1}^{4}(t_{33}^{(k)}+\mu^{(\rho)}u_{\eta,\eta}^{(k)}+C_2^{(\rho)}\varphi_{\eta,\eta}^{(k)})c_k,
\end{aligned}$$

$$
\begin{aligned}
l_\lambda^{(\rho)} &= \varepsilon_{3\lambda\eta}(\gamma^{(\rho)}c_\eta - C_3^{(\rho)}c_4x_\eta) + \sum_{k=1}^{4} c_k\{m_{3\lambda}^{(k)} + \alpha^{(\rho)}\varphi_{3,\lambda}^{(k)} + \\
&\quad + C_1^{(\rho)}u_{3,\lambda}^{(k)} + \varepsilon_{3\lambda\eta}[(C_2^{(\rho)} - C_3^{(\rho)})\varphi_\eta^{(k)} - \kappa u_\eta^{(k)}]\}, \\
l_3^{(\rho)} &= (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})(c_1x_1 + c_2x_2 + c_3) + (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})c_1 + \\
&\quad + \sum_{k=1}^{4} c_k(m_{33}^{(k)} + \beta^{(\rho)}\varphi_{\eta,\eta}^{(k)} + C_2^{(\rho)}u_{\eta,\eta}^{(k)}), \text{ on } \Omega_\rho.
\end{aligned} \tag{35}
$$

On the basis of (21)–(24) and using the conditions (18) which are satisfied by the solution of the problem $A^{(k)}$ $(k = 1,2,3,4)$, we find that the conditions (9) become

$$
\begin{aligned}
&[w_i]_1 = [w_i]_2, \ \ [\chi_i]_1 = [\chi_i]_2, \\
&[\tau_{\beta i}]_1 n_\beta^0 = [\tau_{\beta i}]_2 n_\beta^0 + Y_i^{(1)}, [\mu_{\beta i}]_1 n_\beta^0 = [\mu_{\beta i}]_2 n_\beta^0 + Z_i^{(1)} \ \text{ on } \Gamma,
\end{aligned} \tag{36}
$$

and

$$
\begin{aligned}
&[v_i]_1 = [v_i]_2, \ \ [\psi_i]_1 = [\psi_i]_2, \\
&[\sigma_{\beta i}]_1 n_\beta^0 = [\sigma_{\beta i}]_2 n_\beta^0 + Y_i^{(2)}, [\nu_{\beta i}]_1 n_\beta^0 = [\nu_{\beta i}]_2 n_\beta^0 + Z_i^{(2)} \ \text{ on } \Gamma,
\end{aligned} \tag{37}
$$

In (36) and (37) we have used the notations

$$
\begin{aligned}
Y_\alpha^{(1)} &= \{(\lambda^{(2)} - \lambda^{(1)})v_3 + (C_1^{(2)} - C_1^{(1)})\psi_3 + \sum_{k=1}^{4} b_k[(\lambda^{(2)} - \lambda^{(1)})u_3^{(k)} + \\
&\quad + (C_1^{(2)} - C_1^{(1)})\varphi_3^{(k)}]\}n_\alpha^0, \\
Y_3^{(1)} &= \{(\mu^{(2)} - \mu^{(1)})v_\alpha + (C_2^{(2)} - C_2^{(1)})\psi_\alpha + \\
&\quad + \sum_{k=1}^{4} b_k[(\mu^{(2)} - \mu^{(1)})u_\alpha^{(k)} + (C_2^{(2)} - C_2^{(1)})\varphi_\alpha^{(k)}]\}n_\alpha^0, \\
Z_\alpha^{(1)} &= n_\alpha^0 \sum_{k=1}^{4} c_k[(\lambda^{(2)} - \lambda^{(1)})u_3^{(k)} + (C_1^{(2)} - C_1^{(1)})\varphi_3^{(k)}], \\
Z_3^{(1)} &= \{(\beta^{(2)} - \beta^{(1)})\psi_\alpha + (C_2^{(2)} - C_2^{(1)})v_\alpha + \sum_{k=1}^{4} b_k[(\beta^{(2)} - \beta^{(1)})\psi_\alpha^{(k)} + \\
&\quad + (C_2^{(2)} - C_2^{(1)})u_\alpha^{(k)}]\}n_\alpha^0,
\end{aligned} \tag{38}
$$

and

$$
Y_\alpha^{(2)} = n_\alpha^0 \sum_{k=1}^{4} c_k[(\lambda^{(2)} - \lambda^{(1)})u_3^{(k)} + (C_1^{(2)} - C_1^{(1)})\varphi_3^{(k)}],
$$

$$Y_3^{(2)} = n_\alpha^0 \sum_{k=1}^{4} c_k[(\mu^{(2)} - \mu^{(1)})u_\alpha^{(k)} + (C_2^{(2)} - C_2^{(1)})\varphi_\alpha^{(k)}], \tag{39}$$
$$Z_\alpha^{(2)} = n_\alpha^0 \sum_{k=1}^{4} c_k[(\alpha^{(2)} - \alpha^{(1)})\varphi_3^{(k)} + (C_1^{(2)} - C_1^{(1)})u_3^{(k)}],$$
$$Z_3^{(2)} = n_\alpha^0 \sum_{k=1}^{4} c_k[(\beta^{(2)} - \beta^{(1)})\psi_\alpha^{(k)} + (C_2^{(2)} - C_2^{(1)})u_\alpha^{(k)}].$$

If we use the conditions on L from the problems $A^{(k)}$ $(k = 1,2,3,4)$, then we see that the conditions (4) reduce to

$$[\tau_{\alpha i}]_1 n_\alpha = P_i^{(1)},\ \ [\mu_{\beta i}]_1 n_\beta = Q_i^{(1)} \ \text{ on } L, \tag{40}$$

and

$$[\sigma_{\beta i}]_1 n_\beta = P_i^{(2)},\ \ [\nu_{\beta i}]_1 n_\beta = Q_i^{(2)} \ \text{ on } L. \tag{41}$$

Here we have used the notations

$$P_\alpha^{(1)} = \widetilde{t}_\alpha - [\lambda^{(1)} v_3 + C_1^{(1)}\psi_3 + \sum_{k=1}^{4} b_k(\lambda^{(1)} u_3^{(k)} + C_1^{(1)}\varphi_3^{(k)})]n_\alpha,$$
$$P_3^{(1)} = \widetilde{t}_3 - n_\alpha[\mu^{(1)} v_\alpha + C_2^{(1)}\psi_\alpha + \sum_{k=1}^{4} b_k(\mu^{(1)} u_\alpha^{(k)} + C_2^{(1)}\varphi_\alpha^{(k)})], \tag{42}$$
$$Q_\lambda^{(1)} = \widetilde{m}_\lambda - n_\lambda[\alpha^{(1)}\psi_3 + C_1^{(1)} v_3 + \sum_{k=1}^{4} b_k(\alpha^{(1)}\varphi_3^{(k)} + C_1^{(1)} u_3^{(k)})],$$
$$Q_3^{(1)} = \widetilde{m}_3 - n_\alpha[\beta^{(1)}\psi_\alpha + C_2^{(1)} v_\alpha + \sum_{k=1}^{4} b_k(\beta^{(1)}\psi_\alpha^{(k)} + C_2^{(1)} u_\alpha^{(k)})],$$

and

$$P_\alpha^{(2)} = -n_\alpha \sum_{k=1}^{4} c_k(\lambda^{(1)} u_3^{(k)} + C_1^{(1)}\varphi_3^{(k)}), P_3^{(2)} = -n_\alpha \sum_{k=1}^{4} c_k(\mu^{(1)} u_\alpha^{(k)} + C_2^{(1)}\varphi_\alpha^{(k)}),$$
$$Q_\lambda^{(2)} = -n_\lambda \sum_{k=1}^{4} c_k(\alpha^{(1)}\varphi_3^{(k)} + C_1^{(1)} u_3^{(k)}), Q_3^{(2)} = -n_\alpha \sum_{k=1}^{4} c_k(\beta^{(1)}\psi_\alpha^{(k)} + C_2^{(1)} u_\alpha^{(k)}). \tag{43}$$

Let us denote by (Π_1) the plane strain problem defined by the geometric equations (26), the constitutive equations (27), the equilibrium equations (32) and the conditions (36) and (40). We denote by (Π_2) the plane problem characterized by the geometric equations (28), the constitutive equations (29), the equilibrium equations (33), and the conditions (37) and (41). Let us consider the general plane strain problem defined by the equilibrium equations (14) and the conditions (16) and (18). We establish now an identity which will be used in what follows. In view of (14) we obtain

$$\begin{aligned} t_{3\alpha} &= t_{\alpha 3} + \varepsilon_{3\beta\alpha} m_{\kappa\beta,\kappa} + \varepsilon_{3\beta\alpha} G_\beta^{(\rho)} = x_\alpha (t_{\beta 3,\beta} + F_3^{(\rho)}) + \\ &+ t_{\alpha 3} + \varepsilon_{3\beta\alpha} m_{\kappa\beta,\kappa} + \varepsilon_{3\beta\alpha} G_\beta^{(\rho)} = (x_\alpha t_{\beta 3})_{,\beta} + \varepsilon_{3\beta\alpha} m_{\kappa\beta,\kappa} + \\ &+ \varepsilon_{3\beta\alpha} G_\beta^{(\rho)} + x_\alpha F_3^{(\rho)}, \ \text{ on } \Omega_\rho. \end{aligned} \tag{44}$$

By using the divergence theorem and the relations (16) and (18), from (44) we get

$$\begin{aligned} \int_{\Sigma_1} t_{3\alpha} \mathrm{d}a &= \sum_{\rho=1}^{2} \int_{\Omega_\rho} (x_\alpha F_3^{(\rho)} + \varepsilon_{3\beta\alpha} G_\beta^{(\rho)}) \mathrm{d}a + \int_L (x_\alpha \widetilde{t}_3 + \varepsilon_{3\beta\alpha} \widetilde{m}_\beta) \mathrm{d}s + \\ &+ \int_\Gamma (x_\alpha p_3 + \varepsilon_{3\beta\alpha} q_\beta) \mathrm{d}s. \end{aligned} \tag{45}$$

If we apply the relation (45) to the problems $A^{(k)}$ $(k = 1,2,3,4)$, and use (21)–(24), then we obtain

$$\begin{aligned} &\int_{\Sigma_1} t_{3\alpha}^{(\kappa)} \mathrm{d}a = \sum_{\rho=1}^{2} \int_{\Omega_\rho} \varepsilon_{3\kappa\alpha} C_3^{(\rho)} \mathrm{d}a, \\ &\int_{\Sigma_1} t_{3\alpha}^{(3)} \mathrm{d}a = 0, \ \int_{\Sigma_1} t_{3\alpha}^{(4)} \mathrm{d}a = \sum_{\rho=1}^{2} \int_{\Omega_\rho} (\mu^{(\rho)} + \kappa^{(\rho)}) \varepsilon_{3\alpha\eta} x_\eta \mathrm{d}a. \end{aligned} \tag{46}$$

The necessary and sufficient conditions to solve the problem (Π_2) are given by

$$\begin{aligned} &\sum_{\rho=1}^{2} \int_{\Omega_\rho} h_i^{(\rho)} \mathrm{d}a + \int_L P_i^{(2)} \mathrm{d}s + \int_\Gamma Y_i^{(2)} \mathrm{d}s = 0, \\ &\sum_{\rho=1}^{2} \int_{\Omega_\rho} (\varepsilon_{3\alpha\beta} x_\alpha h_\beta^{(\rho)} + l_3^{(\rho)}) \mathrm{d}a + \int_L (\varepsilon_{3\alpha\beta} x_\alpha P_\beta^{(2)} + Q_3^{(2)}) \mathrm{d}s + \\ &+ \int_\Gamma (\varepsilon_{3\alpha\beta} x_\alpha Y_\beta^{(2)} + Z_3^{(2)}) \mathrm{d}s = 0. \end{aligned} \tag{47}$$

If we use the relations (35), (39) and (43), then we obtain

$$\begin{aligned} \sum_{\rho=1}^{2} \int_{\Omega_\rho} h_\alpha^{(\rho)} \mathrm{d}a + \int_L P_\alpha^{(2)} \mathrm{d}s + \int_\Gamma Y_\alpha^{(2)} \mathrm{d}s &= \sum_{\rho=1}^{2} \int_{\Omega_\rho} \{ c_\kappa (t_{3\alpha}^{(\kappa)} + \varepsilon_{3\alpha\kappa} C_3^{(\rho)}) + c_3 t_{3\alpha}^{(3)} + \\ &+ c_4 [t_{3\alpha}^{(4)} + (\mu^{(\rho)} + \kappa^{(\rho)}) \varepsilon_{3\eta\alpha} x_\eta] \} \mathrm{d}a, \end{aligned}$$

$$\sum_{\rho=1}^{2}\int_{\Omega_\rho} h_3^{(\rho)}\mathrm{d}a+\int_L P_3^{(2)}\mathrm{d}s+\int_\Gamma Y_3^{(2)}\mathrm{d}s=\sum_{k=1}^{4} D_{3k}c_k, \tag{48}$$

$$\sum_{\rho=1}^{2}\int_{\Omega_\rho}(\varepsilon_{3\alpha\beta}x_\alpha h_\beta^{(\rho)}+l_3^{(\rho)})\mathrm{d}a+\int_L(\varepsilon_{3\alpha\beta}x_\alpha P_\beta^{(2)}+Q_3^{(2)})\mathrm{d}s+$$

$$+\int_\Gamma(\varepsilon_{3\alpha\beta}x_\alpha Y_\beta^{(2)}+Z_3^{(2)})\mathrm{d}s=\sum_{k=1}^{4} D_{4k}c_k,$$

where

$$\begin{aligned}
D_{3\alpha}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}[(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})x_\alpha+t_{33}^{(\alpha)}]\mathrm{d}a,\\
D_{33}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)}+t_{33}^{(3)})\mathrm{d}a,\\
D_{34}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)}+t_{33}^{(4)})\mathrm{d}a,\\
D_{4\alpha}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}[\varepsilon_{3\lambda\beta}x_\lambda(C_3^{(\rho)}\varepsilon_{3\beta\alpha}+t_{3\beta}^{(\alpha)})+\\
&\quad+(C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)})x_\alpha+m_{33}^{(\alpha)}]\mathrm{d}a,\\
D_{43}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}[\varepsilon_{3\alpha\beta}x_\alpha t_{3\beta}^{(3)}+C_1^{(\rho)}+C_2^{(\rho)}+C_3^{(\rho)}+m_{33}^{(3)}]\mathrm{d}a,\\
D_{44}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}\{\varepsilon_{3\alpha\beta}x_\alpha[(\mu^{(\rho)}+\kappa^{(\rho)})\varepsilon_{3\eta\beta}x_\eta+t_{3\beta}^{(4)}]+\\
&\quad+\alpha^{(\rho)}+\beta^{(\rho)}+\gamma^{(\rho)}+m_{33}^{(4)}\}\mathrm{d}a.
\end{aligned} \tag{49}$$

We note that the first two conditions from (47) are satisfied on the basis of the relations (46). Thus, the conditions (47) reduce to

$$\sum_{k=1}^{4} D_{3k}c_k=0,\ \ \sum_{k=1}^{4} D_{4k}c_k=0. \tag{50}$$

The necessary and sufficient conditions to solve the problem (Π_1) are

$$\begin{aligned}
&\sum_{\rho=1}^{2}\int_{\Omega_\rho} f_i^{(\rho)}\mathrm{d}a + \int_L P_i^{(1)}\mathrm{d}s + \int_\Gamma Y_i^{(1)}\mathrm{d}s = 0,\\
&\sum_{\rho=1}^{2}\int_{\Omega_\rho}(\varepsilon_{3\alpha\beta}x_\alpha f_\beta^{(\rho)} + g_3^{(\rho)})\mathrm{d}a + \int_L(\varepsilon_{3\alpha\beta}x_\alpha P_\beta^{(1)} + Q_3^{(1)})\mathrm{d}s +\\
&\qquad + \int_\Gamma(\varepsilon_{3\alpha\beta}x_\alpha Y_\beta^{(1)} + Z_3^{(1)})\mathrm{d}s = 0.
\end{aligned} \tag{51}$$

By using the relations (34), (42) and (38) we get

$$\begin{aligned}
&\sum_{\rho=1}^{2}\int_{\Omega_\rho} f_\alpha^{(\rho)}\mathrm{d}a + \int_L P_\alpha^{(1)}\mathrm{d}s + \int_\Gamma Y_\alpha^{(1)}\mathrm{d}s =\\
&\qquad = \int_L \widetilde{t}_\alpha\mathrm{d}s + \int_{\Sigma_1} t_{3\alpha,3}\mathrm{d}a + \sum_{\rho=1}^{2}\int_{\Omega_\rho} F_\alpha^{(\rho)}\mathrm{d}a,
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\sum_{\rho=1}^{2}\int_{\Omega_\rho} f_3^{(\rho)}\mathrm{d}a + \int_L P_3^{(1)}\mathrm{d}s + \int_\Gamma Y_3^{(1)}\mathrm{d}s = \int_L \widetilde{t}_3\mathrm{d}s + \sum_{k=1}^{4} D_{3k}b_k +\\
&\qquad + \int_{\Sigma_1}\sigma_{33}\mathrm{d}a + \sum_{\rho=1}^{2}\int_{\Omega_\rho}\{F_3^{(\rho)} + \sum_{k=1}^{4} c_k[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})u_3^{(k)} +\\
&\qquad + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\varphi_3^{(k)}]\}\mathrm{d}a,\\
&\sum_{\rho=1}^{2}\int_{\Omega_\rho}(\varepsilon_{3\alpha\beta}x_\alpha f_\beta^{(\rho)} + g_3^{(\rho)})\mathrm{d}a + \int_L(\varepsilon_{3\alpha\beta}x_\alpha P_\beta^{(1)} + Q_3^{(1)})\mathrm{d}s +\\
&\qquad + \int_\Gamma(\varepsilon_{3\alpha\beta}x_\alpha Y_\beta^{(1)} + Z_3^{(1)})\mathrm{d}s = \int_L(\varepsilon_{3\alpha\beta}x_\alpha\widetilde{t}_\beta + \widetilde{m}_3)\mathrm{d}s +\\
&\qquad + \sum_{\rho=1}^{2}\int_{\Omega_\rho}(\varepsilon_{3\alpha\beta}x_\alpha F_\beta^{(\rho)} + G_3^{(\rho)})\mathrm{d}a + \sum_{k=1}^{4} D_{4k}b_k +\\
&\qquad + \sum_{\rho=1}^{2}\int_{\Omega_\rho}\{\varepsilon_{3\alpha\beta}x_\alpha\sigma_{3\beta} + \nu_{33} + \sum_{k=1}^{4} c_k[\varepsilon_{3\alpha\beta}(\mu^{(\rho)} + \kappa^{(\rho)})x_\alpha u_\beta^{(k)} +\\
&\qquad + \varepsilon_{3\alpha\beta}C_3^{(\rho)}x_\alpha\varphi_\beta^{(k)} + (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})\varphi_3^{(k)} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})u_3^{(k)}]\}\mathrm{d}a.
\end{aligned} \tag{53}$$

With the help of equilibrium equations (3) and using the conditions (4), (9) and (10) we find that

$$\int_{\Sigma_1} t_{3\alpha}\mathrm{d}a = \int_{\Sigma_1}(x_\alpha t_{33,3} + \varepsilon_{3\beta\alpha} m_{3\beta,3})\mathrm{d}a + \int_L (x_\alpha \widetilde{t}_3 + \varepsilon_{3\beta\alpha}\widetilde{m}_\beta)\mathrm{d}s + \\ + \sum_{\rho=1}^{2}\int_{\Omega_\rho}(x_\alpha F_3^{(\rho)} + \varepsilon_{3\beta\alpha} G_\beta^{(\rho)})\mathrm{d}a, \tag{54}$$

$$\int_{\Sigma_1} t_{3\alpha,3}\mathrm{d}a = \int_{\Sigma_1}(x_\alpha t_{33,33} + \varepsilon_{3\beta\alpha} m_{3\beta,33})\mathrm{d}a.$$

It follows from (30), (31) and (54) that

$$\int_{\Sigma_1} t_{3\alpha}\mathrm{d}a = \int_L (x_\alpha \widetilde{t}_3 + \varepsilon_{3\beta\alpha}\widetilde{m}_\beta)\mathrm{d}s + \sum_{k=1}^{4} D_{\alpha k} b_k + \\ + \sum_{\rho=1}^{2}\int_{\Omega_\rho}\{x_\alpha F_3^{(\rho)} + \varepsilon_{3\beta\alpha} G_\beta^{(\rho)} + x_\alpha \sigma_{33} + \varepsilon_{3\beta\alpha}\nu_{3\beta} + \\ + \sum_{k=1}^{4} c_k[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})x_\alpha u_3^{(k)} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})x_\alpha \varphi_3^{(k)} + \\ + \varepsilon_{3\lambda\alpha}(\gamma^{(\rho)}\psi_\lambda^{(k)} + C_3^{(\rho)} u_\lambda^{(k)})]\}\mathrm{d}a, \tag{55}$$

$$\int_{\Sigma_1} t_{3\alpha,3}\mathrm{d}a = \sum_{k=1}^{4} D_{\alpha k} c_k,$$

where

$$D_{\alpha\beta} = \sum_{\rho=1}^{2}\int_{\Omega_\rho}\{x_\alpha[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})x_\beta + t_{33}^{(\beta)}] + \\ + \gamma^{(\rho)}\delta_{\alpha\beta} + \varepsilon_{3\eta\alpha} m_{3\eta}^{(\beta)}\}\mathrm{d}a,$$

$$D_{\alpha 3} = \sum_{\rho=1}^{2}\int_{\Omega_\rho}[x_\alpha(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)} + t_{33}^{(3)}) + \varepsilon_{3\beta\alpha} m_{3\beta}^{(3)}]\mathrm{d}a, \tag{56}$$

$$D_{\alpha 4} = \sum_{\rho=1}^{2}\int_{\Omega_\rho}[x_\alpha(C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)} + t_{33}^{(4)}) - C_3^{(\rho)} x_\alpha + \varepsilon_{3\beta\alpha} m_{3\beta}^{(4)}]\mathrm{d}a.$$

By (52) and (55) the first two conditions from (51) become

$$\sum_{k=1}^{4} D_{\alpha k} c_k = -\int_L \widetilde{t}_\alpha \mathrm{d}s - \sum_{\rho=1}^{2}\int_{\Omega_\rho} F_\alpha^{(\rho)}\mathrm{d}a. \tag{57}$$

The equations (50) and (57) form a system for the unknown constants c_k. The positive definiteness of the elastic potentials and the reciprocal theorem imply that [18]

$$\det(D_{mn}) > 0, \;\; D_{mn} = D_{nm}. \tag{58}$$

Thus, the system (50), (57) uniquely determines the constants c_k. We note that the conditions (48) are satisfied. In what follows we shall consider that the functions v_j and ψ_j are determined. In view of (53) and (54) we see that the last two conditions from (51) and the conditions (5) become

$$\begin{aligned}
\sum_{k=1}^{4} D_{\alpha k} b_k &= -R_\alpha - \int_L (x_\alpha \widetilde{t}_3 + \varepsilon_{3\beta\alpha}\widetilde{m}_\beta)\mathrm{d}s - \\
&\quad - \sum_{\rho=1}^{2}\int_{\Omega_\rho}\{x_\alpha F_3^{(\rho)} + \varepsilon_{3\beta\alpha}G_\beta^{(\rho)} + x_\alpha \sigma_{33} + \sigma_{3\beta\alpha}\nu_{3\beta} + \\
&\quad + \sum_{k=1}^{4} c_k[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})x_\alpha u_3^{(k)} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})x_\alpha \varphi_3^{(k)} + \\
&\quad + \varepsilon_{3\lambda\alpha}(\gamma^{(\rho)}\psi_\lambda^{(k)} + C_3^{(\rho)}u_\lambda^{(k)})]\}\mathrm{d}a, \\
\sum_{k=1}^{4} D_{3k} b_k &= -\int_L \widetilde{t}_3\mathrm{d}s - \sum_{\rho=1}^{2}\int_{\Omega_\rho}\{F_3^{(\rho)} + \sigma_{33} + \\
&\quad + \sum_{k=1}^{4} c_k[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})u_3^{(k)} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\varphi_3^{(k)}]\}\mathrm{d}a, \\
\sum_{k=1}^{4} D_{4k} b_k &= -\int_L (\varepsilon_{3\alpha\beta}x_\alpha \widetilde{t}_\beta + \widetilde{m}_3)\mathrm{d}s - \sum_{\rho=1}^{2}\{\varepsilon_{3\alpha\beta}x_\alpha F_\beta^{(\rho)} + \\
&\quad + G_3^{(\rho)} + \varepsilon_{3\alpha\beta}x_\alpha\sigma_{3\beta} + \nu_{33} + \sum_{k=1}^{4} c_k[\varepsilon_{3\alpha\beta}(\mu^{(\rho)} + \kappa^{(\rho)})x_\alpha u_\beta^{(k)} + \\
&\quad + \varepsilon_{3\alpha\beta}C_3^{(\rho)}x_\alpha\varphi_\beta^{(k)} + (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})\varphi_3^{(k)} + \\
&\quad + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})u_3^{(k)}]\}\mathrm{d}a.
\end{aligned} \tag{59}$$

The constants b_j $(j = 1,2,3,4)$, can be determined from the system (59). It follows from (30), (31), (49) and (56) that the conditions (6)–(8) reduce to the following system for the constants $a_j\,(j = 1,2,3,4)$,

$$\begin{aligned}
&\sum_{k=1}^{4} D_{\alpha k} a_k = \varepsilon_{3\alpha\beta}(M_\beta + M_\beta^*), \\
&\sum_{k=1}^{4} D_{3k} a_k = -R_3 - R_3^*, \; \sum_{k=1}^{4} D_{4k} a_k = -M_3 - M_3^*,
\end{aligned} \tag{60}$$

where

$$
\begin{aligned}
M_\alpha^* = {} & \sum_{\rho=1}^{2} \int_{\Omega_\rho} \varepsilon_{3\alpha\lambda}\{x_\lambda[\tau_{33} + (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})v_3 + \\
& + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\psi_3] + x_\lambda \sum_{k=1}^{4} b_k[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})u_3^{(k)} + \\
& + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\varphi_3^{(k)}] + \varepsilon_{3\beta\lambda}[\mu_{3\beta} + \gamma^{(\rho)}\psi_\beta + C_3^{(\rho)}v_\beta + \\
& + \sum_{k=1}^{4} b_k(\gamma^{(\rho)}\psi_\beta^{(k)} + C_3^{(\rho)}u_\beta^{(k)})]\}\mathrm{d}a, \\
R_3^* = {} & \sum_{\rho=1}^{2} \int_{\Omega_\rho} \{\tau_{33} + (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})v_3 + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\psi_3 \\
& + \sum_{k=1}^{4} b_k[(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})u_3^{(k)} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})\varphi_3^{(k)}]\}\mathrm{d}a, \\
M_3^* = {} & \sum_{\rho=1}^{2} \int_{\Omega_\rho} \{\varepsilon_{3\lambda\eta}x_\lambda[\tau_{3\eta} + (\mu^{(\rho)} + \kappa^{(\rho)})v_\eta + C_3^{(\rho)}\psi_\eta] + \\
& + \mu_{33} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})v_3 + (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})\psi_3 + \\
& + \varepsilon_{3\lambda\eta}x_\lambda \sum_{k=1}^{4} b_k[(\mu^{(\rho)} + \kappa^{(\rho)})u_\eta^{(k)} + C_3^{(\rho)}\varphi_\eta^{(k)}] + \\
& + \sum_{k=1}^{4} b_k[(\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})\varphi_3^{(k)} + (C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)})u_3^{(k)}]\}\mathrm{d}a.
\end{aligned}
\tag{61}
$$

The system (60) determines the constants $a_j\ (j = 1,2,3,4)$. The three-dimensional problem has been reduced to the study of plane problems. First, we have to determine the solutions of the problems $A^{(k)}\ (k = 1,2,3,4)$ and to calculate the constants $D_{mn}\ (m,n = 1,2,3,4)$. Then, from (50) and (57) we determine the constants $c_j\ (j = 1,2,3,4)$. Next, we determine the solution of the problem (Π_2) and calculate the constants b_j from (59). As the functions $f_j^{(\rho)}, g_j^{(\rho)}, Y_j^{(1)}, Z_j^{(1)}, P_j^{(1)}$ and $Q_j^{(1)}$ are known, we can determine the solution of the problem (Π_1). Then we calculate the constants a_k from (60). It follows from (59), (60) and (61) that the body loads and the tractions on the lateral surface produce extension, bending, torsion and flexure.

If $F_i^{(\rho)} = 0, G_i^{(\rho)} = 0, \widetilde{t}_k = 0$ and $\widetilde{m}_k = 0$, then from the system (50) and (57) we find that $c_j = 0$. The system (59) becomes

$$\sum_{k=1}^{4} D_{\alpha k} b_k = -R_\alpha, \quad \sum_{k=1}^{4} D_{mk} b_k = 0 \quad (m = 3,4).$$

From (60) and (61) we see that the flexure of chiral cylinders produces extension, bending and torsion. In the case of achiral materials the flexure is not accompanied by extension and bending by terminal couples. The problem of chiral homogeneous cylinders was investigated in [24].

5. REINFORCED CIRCULAR CYLINDER SUBJECTED TO A LATERAL PRESSURE

In this section we use the solution (25) to investigate the deformation of a uniformly loaded circular cylinder reinforced by a longitudinal rod. We assume that the domain Ω_1 is bounded by two concentric circles Γ and L, of radius r_2 and r_1, respectively, where $r_2 < r_1$. The domain Ω_2 is bounded by the circle Γ. We suppose that the cylinders B_1 and B_2 are occupied by two different isotropic and homogenenous materials. We assume that the cylinder is subjected to the following system of loads

$$F_i^{(\rho)} = 0, G_i^{(\rho)} = 0, \widetilde{t}_\alpha = Pn_\alpha, \widetilde{t}_3 = 0, \widetilde{m}_j = 0, R_j = 0, M_j = 0, \tag{62}$$

where P is a given constant. In this case, from the system (50), (57) we obtain $c_j = 0$ $(j = 1,2,3,4)$. Then, the relations (35), (39) and (43) imply that $h_j^{(\rho)} = 0, l_j^{(\rho)} = 0$, $Y_j^{(2)} = 0$, $Z_j^{(2)} = 0, P_j^{(2)} = 0$ and $Q_j^{(2)} = 0$. Thus, the problem (Π_2) has the solution $v_k = 0$ and $\psi_k = 0$. This fact implies that $\sigma_{3k} = 0$ and $\mu_{3k} = 0$. From (59) and (62) we find that $b_j = 0$ $(j = 1,2,3,4)$. Let us study now the problem (Π_1). We note that in this case the relations (34), (38), (42) and (62) imply that

$$f_j^{(\rho)} = 0, g_j^{(\rho)} = 0, Y_j^{(1)} = 0, Z_j^{(1)} = 0, P_\alpha^{(1)} = Pn_\alpha, P_3^{(1)} = 0, Q_j^{(1)} = 0. \tag{63}$$

In view of relations (26) and (27) we can express the equilibrium equations (32) in the form (20).

We seek the solution of the problem (Π_1) in the form

$$w_\alpha = x_\alpha F(r),\ w_3 = 0, \chi_\alpha = x_\alpha G(r),\ \ \chi_3 = 0, \tag{64}$$

where F and G are unknown functions and $r = (x_1^2 + x_2^2)^{1/2}$. We introduce the notations

$$\begin{aligned}
&\xi_1^{(\rho)} = \lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}, \xi_2^{(\rho)} = 2\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)},\\
&\xi_3^{(\rho)} = \alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}, \xi_4^{(\rho)} = 2\mu^{(\rho)} + \kappa^{(\rho)},\\
&\zeta_1^{(\rho)} = C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)}, \zeta_2^{(\rho)} = 2C_1^{(\rho)} + C_2^{(\rho)} + C_3^{(\rho)},\\
&\zeta_3^{(\rho)} = C_2^{(\rho)} + C_3^{(\rho)},\\
&s_{(\rho)}^2 = 2\kappa^{(\rho)}/\xi_3^{(\rho)}, q_{(\rho)}^2 = 2\kappa^{(\rho)}\xi_1^{(\rho)}[\xi_1^{(\rho)}\xi_3^{(\rho)} - (\zeta_1^{(\rho)})^2]^{-1},\\
&\nu^{(\rho)} = \zeta_1^{(\rho)}/\xi_1^{(\rho)}.
\end{aligned} \tag{65}$$

The positive definiteness of elastic potential implies that $q_{(\rho)}^2 > 0$. The functions F and G which satisfy the equations (32) are given by

$$F = F_1(r), G = G_1(r) \text{ on } \Omega_1, F = F_2(r), G = G_2(r) \text{ on } \Omega_2, \tag{66}$$

where

$$\begin{aligned}
&F_1 = D_1^{(1)} + D_2^{(1)} r^{-2} - \nu^{(1)} r^{-1}[A_1^{(1)} I_1(q_{(1)}r) + A_2^{(1)} K_1(q_{(1)}r)],\\
&G_1 = r^{-1}[A_1^{(1)} I_1(q_{(1)}r) + A_2^{(1)} K_1(q_{(1)}r)],\\
&F_2 = D_1^{(2)} - \nu^{(2)} r^{-1} A_1^{(2)} I_1(q_{(2)}r), G_2 = r^{-1} A_1^{(2)} I_1(q_{(2)}r).
\end{aligned} \tag{67}$$

Here, I_n and K_n are the modified Bessel functions of order n, and $A_\alpha^{(\rho)}$ and $D_\alpha^{(\rho)}$ are arbitrary constants. Let us study now the conditions (36) and (40). We note that on the boundary L we have $n_\alpha = x_\alpha/r_1$, and on the boundary Γ we have $n_\alpha^0 = -x_\alpha/r_2$. In view of (27) and (65) we obtain

$$\begin{aligned}
&\tau_{\alpha\beta}x_\alpha = x_\beta[\xi_2^{(\rho)} F + \xi_1^{(\rho)} rF' + \zeta_2^{(\rho)} G + \zeta_1^{(\rho)} rG'],\\
&\mu_{\alpha\beta}x_\alpha = x_\beta[(\alpha^{(\rho)} + \xi_3^{(\rho)})G + \xi_3^{(\rho)} rG' + \zeta_2^{(\rho)} F + \zeta_1^{(\rho)} rF'],\\
&\tau_{\alpha 3}x_\alpha = 0,\ \mu_{\alpha 3}x_\alpha = 0,\ \text{ on } \Omega_\rho\ (\rho = 1,2).
\end{aligned} \tag{68}$$

We introduce the notations

$$\begin{aligned}
&E^{(\rho)} = C_1^{(\rho)} - \nu^{(\rho)}\lambda^{(\rho)},\ J^{(\rho)} = \zeta_1^{(\rho)} - \nu^{(\rho)}\xi_1^{(\rho)},\\
&R^{(\rho)} = \alpha^{(\rho)} - C_1^{(\rho)}\nu^{(\rho)}, L^{(\rho)} = \xi_3^{(\rho)} - \nu^{(\rho)}\zeta_1^{(\rho)},\\
&\Lambda^{(\rho)}(r) = E^{(\rho)} r^{-1} I_1(q_{(\rho)}r) + J^{(\rho)} I_1'(q_{(\rho)}r), H^{(\rho)}(r) = E^{(\rho)} r^{-1} K_1(q_{(\rho)}r) +\\
&\qquad + J^{(\rho)} K_1'(q_{(\rho)}r), V^{(\rho)}(r) = R^{(\rho)} r^{-1} I_1(q_{(\rho)}r) + L^{(\rho)} I_1'(q_{(\rho)}r),\\
&U^{(\rho)}(r) = R^{(\rho)} r^{-1} K_1(q_{(\rho)}r) + L^{(\rho)} K_1'(q_{(\rho)}r).
\end{aligned} \tag{69}$$

By using (66), (67) and (69), from (68) we get

$$\begin{aligned}
&\tau_{\alpha\beta}x_\alpha = x_\beta[\xi_2^{(1)}D_1^{(1)} - \xi_4^{(1)}r^{-2}D_2^{(1)} + \Lambda^{(1)}(r)A_1^{(1)} + H^{(1)}(r)A_2^{(1)}],\\
&\mu_{\alpha\beta}x_\alpha = x_\beta[\zeta_2^{(1)}D_1^{(1)} - \zeta_3^{(1)}D_2^{(1)}r^{-2} + V^{(1)}(r)A_1^{(1)} + U^{(1)}(r)A_2^{(1)}], \text{ on } \Omega_1,
\end{aligned} \tag{70}$$

and

$$\begin{aligned}
&\tau_{\alpha\beta}x_\alpha = x_\beta[\xi_2^{(2)}D_1^{(2)} + \Lambda^{(2)}(r)A_1^{(2)}],\\
&\mu_{\alpha\beta}x_\alpha = x_\beta[\zeta_2^{(2)}D_1^{(2)} + V^{(2)}(r)A_1^{(2)}], \text{ on } \Omega_2.
\end{aligned} \tag{71}$$

In view of (64), (66), (67), (70) and (71) the conditions (36) and (40) reduce to

$$\begin{aligned}
&D_1^{(1)} + D_2^{(1)}r^{-2} - \nu^{(1)}r_2^{-1}[A_1^{(1)}I_1(q_{(1)}r_2) + A_2^{(1)}K_1(q_{(1)}r_2)] =\\
&\qquad = D_1^{(2)} - \nu^{(2)}r_2^{-1}A_1^{(2)}I_1(q_{(2)}r_2),\\
&A_1^{(1)}I_1(q_{(1)}r_2) + A_2^{(1)}K_1(q_{(1)}r_2) = A_1^{(2)}I_1(q_{(2)}r_2),\\
&\xi_2^{(1)}D_1^{(1)} - \xi_4^{(1)}r_2^{-2}D_2^{(1)} + \Lambda^{(1)}(r_2)A_1^{(1)} + H^{(1)}(r_2)A_2^{(1)} =\\
&\qquad = \xi_2^{(2)}D_1^{(2)} + \Lambda^{(2)}(r_2)A_1^{(2)},\\
&\zeta_2^{(1)}D_1^{(1)} - \zeta_3^{(1)}D_2^{(1)}r_2^{-2} + V^{(1)}(r_2)A_1^{(1)} + U^{(1)}(r_2)A_2^{(1)} =\\
&\qquad = \zeta_2^{(2)}D_1^{(2)} + V^{(2)}(r_2)A_1^{(2)},\\
&\xi_2^{(1)}D_1^{(1)} - \xi_4^{(1)}r_1^{-2}D_2^{(1)} + \Lambda^{(1)}(r_1)A_1^{(1)} + H^{(1)}(r_2)A_2^{(1)} = P,\\
&\zeta_2^{(1)}D_1^{(1)} - \zeta_3^{(1)}D_2^{(1)}r_1^{-2} + V^{(1)}(r_1)A_1^{(1)} + U^{(1)}(r_1)A_2^{(1)} = 0.
\end{aligned} \tag{72}$$

We assume that the constitutive coefficients are not subjected to constraints. From the system (72) we can determine the unknown constants $A_\alpha^{(1)}, A_1^{(2)}, D_\alpha^{(1)}$ and $D_1^{(2)}$. In a similar way we can solve the problems $A^{(3)}$ and $A^{(4)}$ [14]. The solution of the problem $A^{(3)}$ has the form

$$u_\alpha^{(3)} = x_\alpha U(r), u_3^{(3)} = 0, \varphi_\alpha^{(3)} = x_\alpha V(r), \varphi_3^{(3)} = 0, \tag{73}$$

where the functions U and V are given by

$$\begin{aligned}
&U = N_1 + N_2r^{-2} - \nu^{(1)}r^{-1}[S_2I_1(q_{(1)}r) + S_3K_1(q_{(1)}r)],\\
&V = r^{-1}[S_2I_1(q_{(1)}r) + S_3K_1(q_{(1)}r)], \text{ on } \Omega_1,\\
&U = S_1 - \nu^{(2)}r^{-1}S_4I_1(q_{(2)}r), V = r^{-1}S_4I_1(q_{(2)}r), \text{ on } \Omega_2,
\end{aligned} \tag{74}$$

where N_α and S_j $(j = 1,2,3,4)$ are constants. The constants S_j are determined by the system

$$\sum_{j=1}^{4} M_{kj}S_j = c_k^* \quad (k = 1,2,3,4), \tag{75}$$

where M_{rs} and c_k^* are given by

$$\begin{aligned}
&M_{11} = 1 - q_{11} - q_{21}r_2^{-2},\ M_{12} = -q_{12} - q_{22}r_2^{-2},\ \ M_{13} = -q_{13} - q_{23}r_2^{-2},\\
&M_{14} = -q_{24}r_2^{-2} - \nu^{(2)}r_2^{-1}I_1(q_{(2)}r_2),\ M_{21} = 0,\ M_{22} = I_1(q_{(2)}r_2),\\
&M_{23} = K_1(q_{(1)}r_2),\ M_{24} = -I_1(q_{(2)}r_2),\\
&M_{31} = \zeta_2^{(1)}q_{11} - \zeta_3^{(1)}r_2^{-2}q_{21} - \zeta^{(2)},\ \ M_{32} = \zeta_2^{(1)}q_{12} - \zeta_3^{(1)}q_{22}r_2^{-2} + V^{(1)}(r_2),\\
&M_{33} = \zeta_2^{(1)}q_{13} - \zeta_3^{(1)}r_2^{-2}q_{23} + U^{(1)}(r_2),\ \ M_{34} = -V^{(2)}(r_2),\\
&M_{41} = \zeta_2^{(1)}q_{11} - \zeta_3^{(1)}r_1^{-2}q_{21},\ \ M_{42} = \zeta_2^{(1)}q_{12} - \zeta_3^{(1)}r_1^{-2}q_{22} + V^{(1)}(r_1),\\
&M_{43} = \zeta_2^{(1)}q_{13} - \zeta_3^{(1)}r_1^{-2}q_{23} + U^{(1)}(r_1),\ \ M_{44} = \zeta_2^{(1)}q_{14} - \zeta_3 r_1^{-2}q_{24}, q_{11} = \xi_2^{(2)}r_1^{-2}d_1,
\end{aligned}$$

$$\begin{aligned}
&q_{12} = [\Lambda^{(1)}(r_1)r_2^{-2} - \Lambda^{(1)}(r_2)r_1^{-2}]d_1,\ q_{13} = [H^{(1)}(r_1)r_2^{-2} - H^{(1)}(r_2)r_1^{-2}]d_1,\\
&q_{14} = \Lambda^{(2)}(r_2)r_1^{-2}d_1,\ q_{21} = \xi_2^{(2)}d_2,\ q_{22} = [\Lambda^{(1)}(r_1) - \Lambda^{(1)}(r_2)]d_2,\\
&q_{23} = [H^{(1)}(r_1) - H^{(1)}(r_2)]d_2,\ q_{24} = d_2\Lambda^{(2)}(r_2),\\
&d_1 = [\xi_2^{(1)}(r_1^{-2} - r_2^{-2})]^{-1},\ d_2 = [\xi_4^{(1)}(r_1^{-2} - r_2^{-2})]^{-1},\\
&c_1^* = -c_1^0 - c_2^0 r_2^{-2},\ \ c_2^* = 0,\ \ c_3^* = C_1^{(2)} - C_1^{(1)} - c_1^0\zeta_2^{(1)} + c_2^0\zeta_2^{(1)}r_2^{-2},\\
&c_4^* = \zeta_3^{(1)}c_2^0 r_1^{-2} - c_1^0\zeta_2^{(1)} - C_1^{(1)},\ \ c_1^0 = [(\lambda^{(2)} - \lambda^{(1)})r_1^{-2} + \lambda^{(1)}r_2^{-2}]d_1, c_2^0 = \lambda^{(2)}d_2.
\end{aligned} \tag{76}$$

The constants N_α are given by

$$N_\alpha = c_\alpha^0 + \sum_{k=1}^{4} q_{\alpha k}S_k. \tag{77}$$

The solution of the problem $A^{(4)}$ has the form

$$u_\alpha^{(4)} = x_\alpha\Phi(r), u_3^{(4)} = 0, \varphi_\alpha^{(4)} = x_\alpha[\Psi(r) - \frac{1}{2}], \varphi_3^{(4)} = 0, \tag{78}$$

where

$$\begin{aligned}
&\Phi = Q_1 + Q_2 r^{-2} - \nu^{(1)}r^{-1}[L_2 I_1(q_{(1)}r) + L_3 K_1(q_{(1)}r)],\\
&\Psi = r^{-1}[L_2 I_1(q_{(1)}r) + L_3 K_1(q_{(1)}r)],\ \text{ on } \Omega_1,\\
&\Phi = L_1 - \nu^{(2)}r^{-1}L_4 I_1(q_{(2)}r), \Psi = r^{-1}L_4 I_1(q_{(2)}r),\ \text{on } \Omega_2.
\end{aligned} \tag{79}$$

In (79) Q_α and L_j are constants. From the conditions on the curves L and Γ we find that the constants L_j $(j=1,2,3,4)$, are determined by the following system

$$\sum_{j=1}^{4} M_{kj}L_j = \vartheta_k \ \ (k=1,2,3,4), \tag{80}$$

where

$$\begin{aligned}
&\vartheta_1 = -\sigma_1 - \sigma_2 r_2^{-1}, \vartheta_2 = 0, \vartheta_3 = \frac{1}{2}(\beta^{(1)}+\gamma^{(1)}-\beta^{(2)}-\gamma^{(2)}) - \zeta_2^{(1)}\sigma_1 - \\
&\quad - \zeta_3^{(1)} r_2^{-2}\sigma_2, \vartheta_4 = \frac{1}{2}(\beta^{(1)}+\gamma^{(1)}) - \zeta_2^{(1)}\sigma_1 + \zeta_3^{(1)} r_1^{-2}\sigma_2, \\
&\sigma_1 = \frac{1}{2}[(\zeta_3^{(1)}-\zeta_3^{(2)})r_1^{-2} - \zeta_3^{(1)} r_2^{-1}]d_1, \sigma_2 = -\frac{1}{2}\zeta_3^{(2)} d_2.
\end{aligned} \tag{81}$$

The constants Q_α are given by

$$Q_\alpha = \sigma_\alpha + \sum_{k=1}^{4} q_{\alpha j} L_j. \tag{82}$$

In what follows we shall see that the considered problem can be solved without using the solution of the problems $A^{(1)}$ and $A^{(2)}$. As we know the solutions of the problems $A^{(3)}$ and $A^{(4)}$, from (49) and (56) we can calculate the constants D_{j3} and D_{j4}. We obtain

$$\begin{aligned}
&D_{\alpha 3} = 0, \ \ D_{\alpha 4} = 0, \\
&D_{33} = \pi\xi_1^{(1)}(r_1^2 - r_2^2) + \pi\xi_1^{(2)} r_2^2 + 2N_1\lambda^{(1)}\pi(r_1^2 - r_2^2) + \\
&\quad + 2\pi r_2^2 S_1 \lambda_1^{(2)} + 2\pi(C_1^{(1)} - \nu^{(1)}\lambda^{(1)})[\mathscr{A}_1(r_1) - \mathscr{A}_1(r_2)] + \\
&\quad + 2\pi(C_1^{(2)} - \nu^{(2)}\lambda^{(2)}) S_4 r_2 I_1(q_{(2)} r_2), \\
&D_{34} = \pi(C_2^{(1)} + C_3^{(1)})(r_1^2 - r_2^2) + \pi(C_2^{(2)} + C_3^{(2)}) r_2^2 + 2Q_1\lambda^{(1)}\pi(r_1^2 - r_2^2) + \\
&\quad + 2\pi L_1 \lambda^{(2)} r_2^2 + 2\pi(C_1^{(1)} - \nu^{(1)}\lambda^{(1)})[\mathscr{A}_2(r_1) - \mathscr{A}_2(r_2)] + \\
&\quad + 2\pi(C_1^{(2)} - \nu^{(2)}\lambda^{(2)}) L_4 r_2 I_1(q_{(2)} r_2), \\
&D_{44} = \frac{1}{4}\pi(2\mu^{(1)} + \kappa^{(1)})(r_1^2 - r_2^2) + \frac{1}{4}\pi(2\mu^{(2)} + \kappa^{(2)}) r_2^2 + \\
&\quad + \pi(\beta^{(1)} + \gamma^{(1)})(r_1^2 - r_2^2) + \pi(\beta^{(2)} + \gamma^{(2)}) r_2^2 + 2\pi Q_1 C_1^{(1)}(r_1^2 - r_2^2) + \\
&\quad + 2\pi L_1 C_1^{(2)} r_2^2 + 2\pi\kappa^{(1)}[\mathscr{H}_1(r_1) - \mathscr{H}_1(r_2)] + \\
&\quad + 2\pi(\alpha^{(1)} - \nu^{(1)} C_1^{(1)})[\mathscr{H}_2(r_1) - \mathscr{H}_2(r_2)] + 2\pi\kappa^{(2)} q_{(2)}^{-1} r_2^2 L_2 I_2(q_{(2)} r_2) + \\
&\quad + 2\pi(\alpha^{(2)} - \nu^{(2)} C_1^{(1)}) r_2 L_4 I_1(q_{(2)} r_2),
\end{aligned} \tag{83}$$

where

$$\mathscr{A}_1(r)=r[S_2I_1(q_{(1)}r)+S_3K_1(q_{(1)}r)], \mathscr{A}_2(r)=r[L_2I_1(q_{(1)}r)+L_3K_1(q_{(1)}r)],$$
$$\mathscr{H}_1(r)=q_{(1)}^{-2}r^2[L_2I_2(q_{(1)}r)-L_3K_2(q_{(1)}r)], \tag{84}$$
$$\mathscr{H}_2(r)=r[L_2I_1(q_{(1)}r)+L_3K_1(q_{(1)}r)].$$

From (61) we get

$$\begin{aligned}
R_3^* = {} & 2\pi\{r_2^2[\lambda^{(2)}F_2(r_2)+C_1^{(2)}G_2(r_2)]+r_1^2[\lambda^{(1)}F_1(r_1)+\\
& +C_1^{(1)}G_1(r_1)]-r_2^2[\lambda^{(1)}F_1(r_2)+C_1^{(1)}G_1(r_2)\},\ M_\alpha^*=0,\\
M_3^* = {} & 2\pi A_1^{(2)}q_{(2)}^{-1}\kappa^{(1)}r_2I_2(q_{(2)}r_2)+2\pi\kappa^{(2)}q_{(1)}^{-1}\{A_1^{(1)}[r_1^2I_2(q_{(1)}r_1)-\\
& -r_2^2I_2(q_{(1)}r_2)]-A_2^{(1)}[r_1^2K_2(q_{(1)}r_1)-r_2^2K_2(q_{(1)}r_2)]\}+\\
& +2\pi\{r_2^2[C_1^{(2)}F_2(r_2)+\alpha^{(2)}G_2(r_2)]+r_1^2[C_1^{(1)}F_1(r_1)+\\
& +\alpha^{(1)}G_1(r_1)]-r_2^2[C_1^{(1)}F_1(r_2)+\alpha^{(1)}G_1(r_2)]\}.
\end{aligned} \tag{85}$$

By using the symmetry of D_{mn} and (83) we find that

$$D_{3\alpha}=D_{4\alpha}=0. \tag{86}$$

In view of (83), (85) and (86), the system (60) reduces to

$$D_{\alpha\beta}a_\beta=0,\ D_{33}a_3+D_{34}a_4=-R_3^*,\ D_{34}a_3+D_{44}a_4=-M_3^*. \tag{87}$$

Thus, the constants a_k are given by

$$a_\alpha=0,\ a_3=(D_{34}M_3^*-D_{44}R_3^*)d^{-1}, a_4=(D_{34}R_3^*-D_{33}M_3^*)d^{-1}, \tag{88}$$

where $d=D_{33}D_{44}-D_{34}^2$. From (25) we find that the solution of the problem is given by

$$\begin{aligned}
& u_\alpha=\varepsilon_{3\beta\alpha}a_4x_\beta x_3+x_\alpha(a_3U+a_4\Phi+F),\\
& \varphi_\alpha=x_\alpha[a_3V+a_4(\Psi-\frac{1}{2})+G],\\
& u_3=a_3x_3,\ \ \varphi_3=a_4x_3,
\end{aligned} \tag{89}$$

where the functions F,G,U,V,Φ and Ψ are defined by (66),(67),(74), (75), (79) and the constants a_3 and a_4 are given by (88). Thus, a uniform pressure acting on the lateral surface of the cylinder generates an extension and a torsion. In the case of a cylinder composed by achiral materials we have $C_j^{(\rho)}=0$, $\nu^{(\rho)}=0$, $(\rho=1,2)$ and from (72) we obtain $A_\alpha^{(1)}=0, A_1^{(2)}=0$. Then, from (83) and (85) we find that $D_{34}=0$ and $M_3^*=0$, so that a uniform pressure on the lateral surface of a cylinder composed by achiral materials does not produce torsional effects.

6. CONCLUSIONS

The results established in this paper can be summarized as follows:

(a) We study the deformation of a bar composed by two different chiral elastic materials welded together along the surface of separation. We consider the equilibrium of a bar which is subjected to body loads, to tractions on the lateral surface and to resultant forces and moments on the ends. The problem is reduced to the study of two-dimensional problems.

(b) We present the solution of the problem of loaded cylinders when the body loads and the tractions on the lateral surface are independent of the axial coordinate. We show that the body loads and the tractions on the lateral surface produce extension, bending, torsion and flexure.

(c) We present, as a special case, the solution of the problem of flexure by a transversal force.

(d) We use the method to investigate the behaviour of a circular cylinder reinforced by a longitudinal rod. We show that, in contrast with the case of achiral solids, a uniform pressure on the surface of the cylinder produces torsional effects.

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