# RICCI SOLITON EQUATION WITH APPLICATION TO A COUPLED PENDULA

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*Abstract.* The Ricci equation and the soliton solutions are obtained in this paper for two coupled pendula in order to optimise its locomotion and structure. The robot is swinging repeatedly like a rope with successive movement steps along a mobile support. The cnoidal theory and a genetic algorithm are used to solve the problem via the Ricci solitons and the pseudospherical reduction of the rheological Zener equations. The Bäcklund transform is applied to Ricci equation to generate pseudo-spherical surfaces.

*Key words:* Ricci soliton, pendulum, cnoidal theory.

## 1. INTRODUCTION

The motion of pendulum is discussed in many papers with emphasis to design the controller's structures [1–9]. The structure varies from the classical type with or without feedback linearization to the hybrid neuro-fuzzy controllers. To obtain an appropriate structure of the controller, the design method makes use of tuning, trial and error procedures with various learning and evolutionary methods. Searching for a satisfactory procedure follows the finding and tuning of the multiple parameter characterization of the controllers.

The behavior of pendulum has attracted considerable attention in recent years. The coupled oscillators are used to model physical, chemical and biological systems such as coupled *p*-*n* junctions, Josephson-junction arrays, the charge-density waves, chemical-reaction systems and biological oscillators [10–14]. Bäcklund and Darboux transformations and the geometry and modern applications in soliton theory are presented in [15–17].

Tzitzeica equation is a nonlinear partial differential equation derived by Gheorghe Țițeica in 1907 in describing surfaces of constant affine curvature. Tzitzeica equation has also been used in nonlinear physics, being an integrable 1+1 dimensional Lorentz invariant system. Tzitzeica equation is discussed in [18, 19].

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In this paper we attach to the coupled pendula two sets of equations, namely the exact equations and the simplified equations obtained from the first one by simplification [20]. The cnoidal theory and a genetic algorithm are used to solve the problem. The cnoidal method his new method uses the cnoidal wave as the fundamental basis function [20]. We obtain the Q -function representation of the solutions as a linear superposition of Jacobean elliptic functions (cnoidal vibrations) and additional terms, which include nonlinear interactions among the vibrations. The cnoidal vibrations are much richer than sine vibrations because the modulus m of the cnoidal vibrations $(0 < m < 1)$ can vary to obtain a sine vibration $(m \cong 0)$, a Stokes vibration $(m \cong 0.5)$ or a soliton vibration $(m \cong 1)$.

In this paper, the two coupled pendula is modeled as a forced oscillator equation

$$\dot{h}(t) + \varepsilon h(t) = g(t)\,, \tag{1}$$

where $\varepsilon \in [-1,1]$ and the function $g(t)$ defined as

$$g(t) = -\frac{1}{a\varepsilon t + b},\, a,b \in \mathrm{R},\, a \neq 0,\, \varepsilon = \pm 1\,. \tag{2}$$

The equation (1) can be written as the Ricci equation [6]

$$\dot{h} = Lg + 2Ag\,, \tag{3}$$

with $Lg = g(1-2A) - \varepsilon h$.

We recognize in (3) the Ricci equation [7]. The equation (1) can be rewritten as

$$\frac{1}{\varepsilon f(t) + \ddot{f}(t)} = -(a\varepsilon t + b)\,. \tag{4}$$

By differentiating (4) with respect to time

$$\frac{\varepsilon \dot{f}(t) + \dddot{f}(t)}{[\varepsilon f(t) + \ddot{f}(t)]^2} = a\varepsilon\,, \tag{5}$$

we obtain a relation which is used to relate (1) to the Tzitzeica surface [18–20]

$$\begin{aligned} &ah = h_u,\, a_v = ba'' + h,\, b_v + bb'' = 0, \\ &b''h = h_v,\, a'' + aa'' = 0,\, b''_u + a''b = h, \end{aligned} \tag{6}$$

where $a, a', a''...$ are the centro-affine invariant functions that defines the position vector $r(u,v)$ of a surface $\Sigma$ in $\mathrm{R}^3$,

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k, (u,v) \in D(t), \tag{7}$$

where $D(t)$ is a family of vector fields generated by $\{\psi_t\} = \{u(t), v(t)\}$.
The vector $r(u,v)$, which satisfies the condition

$$(r, r_u, r_v) \neq 0, \tag{8}$$

can be considered the solution of the second-order partial differential equations system that defines a surface leaving a centro-affinity aside

$$r_{uu} = ar_u + br_v + cr, r_{uv} = a'r_u + b'r_v + c'r, r_{vv} = a''r_u + b''r_v + c''r, \tag{9}$$

which is completely integrable, that is

$$(r_{uu})_v = (r_{uv})_u, (r_{uv})_v = (r_{vv})_u, \tag{10}$$

where $a, a', a''...$ are the centroaffine invariant functions of $u$ and $v$. For $c = c'' = 0$, the surface $\Sigma$ is related to the asymptotic lines. Some applications such as a nonlinear system with essential energy influx: the human cardiovascular system, propagation of ultrasonic waves in nonlinear multilayered media and cylindrical Tzitzeica curves with forced harmonic oscillators are discussed in [20–26].

## 2. BASIC EQUATIONS

Fig. 1 shows a coupled pendulum consisted from two straight rods $O_1Q_1$, $O_2Q_2$ of masses $M_1, M_2$, lengths $O_1Q_1 = O_2Q_2 = a$, and mass centres $C_1, C_2$ with $O_1C_1 = l_1, O_2C_2 = l_2$ and $O_1O_2 = l$. The rods are linked together by an elastic spring $Q_1Q_2, Q_1 \in O_1C_1, Q_2 \in O_2C_2$ characterised by an elastic constant $k$. The elastic force in the spring is given by $k\,|O_1O_2 - Q_1Q_2|$.

The kinetic energy $T$ of the system is

$$T = \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2), \tag{11}$$

where $\theta_1$ and $\theta_2$ are the displacement angles in rapport to the verticals, $I_1$ is the mass moment of inertia of $O_1O_2$ with respect to $C_1$ and $I_2$ is the mass moment of inertia of $O_3O_4$ with respect to $C_2$.

The elastic potential is

$$U = g(M_1l_1\cos\theta_1 + M_2l_2\cos\theta_2) - \frac{k}{2}(O_1O_2 - Q_1Q_2)^2, \tag{12}$$

where

$$Q_1Q_2^2 = [O_1O_2 + a(\sin\theta_2 - \sin\theta_1)]^2 + a^2(\cos\theta_2 - \cos\theta_1)^2 = \\ = O_1O_2^2 + 2aO_1O_2(\sin\theta_2 - \sin\theta_1) + 2a^2[1 - \cos(\theta_2 - \theta_1)]. \tag{13}$$

From Lagrange equations we derive the motion equations of the pendulum

$$\begin{cases} I_1\ddot{\theta}_1 + M_1 g l_1 \sin\theta_1 + \dfrac{k}{2}\dfrac{\partial}{\partial\theta_1}(O_1O_2 - Q_1Q_2)^2 = 0, \\ I_2\ddot{\theta}_2 + M_2 g l_2 \sin\theta_2 + \dfrac{k}{2}\dfrac{\partial}{\partial\theta_2}(O_1O_2 - Q_1Q_2)^2 = 0, \end{cases} \tag{14}$$

with $g$ the gravitational acceleration. Equations (14) are coupled and nonlinear. By substituting (13) into (14) we have

$$\begin{cases} I_1\ddot{\theta}_1 + M_1 g l_1 \sin\theta_1 - kH[-al\cos\theta_1 - a^2\sin(\theta_2 - \theta_1)] = 0, \\ I_2\ddot{\theta}_2 + M_2 g l_2 \sin\theta_2 - kH[al\cos\theta_2 + a^2\sin(\theta_2 - \theta_1)] = 0, \end{cases} \tag{15}$$

where

$$\mathrm{H}(\theta_1,\theta_2) = \frac{l - \Psi(\theta_1,\theta_2)}{\Psi(\theta_1,\theta_2)}, \tag{16}$$

$$\Psi(\theta_1,\theta_2) = Q_1Q_2 = \\ = [l^2 + 2al(\sin\theta_2 - \sin\theta_1) + 2a^2(1 - \cos(\theta_2 - \theta_1))]^{1/2}. \tag{17}$$

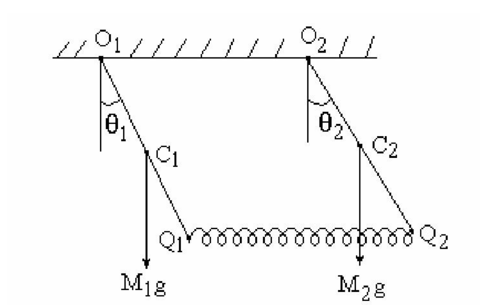


Fig. 1 – Two coupled pendula.

Defining the dimensionless variable $\tau = t\sqrt{\dfrac{k}{M_1}}$ and introducing the notations

$$\frac{M_1^2 g l_1}{I_1 k} = w, \quad \frac{M_1 M_2 g l_2}{I_2 k} = \beta w, \quad \Phi = \frac{\Psi}{l},$$
$$\frac{a}{l} = \xi, \quad \frac{AM_1}{kI_1} = \delta, \quad \frac{alM_1}{I_1} = \alpha, \quad \frac{alM_1}{I_2} = \tilde{\alpha}. \tag{18}$$

The equations (15) are reduced to the dimensionless equations

$$\begin{cases} \ddot{\theta}_1 + w\sin\theta_1 + \gamma(\theta_1,\theta_2)\alpha[\cos\theta_1 + \xi\sin(\theta_2-\theta_1)] = 0, \\ \ddot{\theta}_2 + \beta w\sin\theta_2 - \gamma(\theta_1,\theta_2)\tilde{\alpha}[\cos\theta_2 + \xi\sin(\theta_2-\theta_1)] = 0, \end{cases} \tag{19}$$

where $"\cdot"$ means the differentiation with respect to $\tau$ and

$$\gamma(\theta_1,\theta_2) = \Phi^{-1/2} - 1,$$
$$\Phi(\theta_1,\theta_2) = 1 + 2\xi(\sin\theta_2 - \sin\theta_1) + 2\xi^2(1-\cos(\theta_2-\theta_1)). \tag{20}$$

The associated initial conditions are

$$\theta_1(0) = \theta_1^0, \;\; \theta_2(0) = \theta_2^0, \;\; \dot{\theta}_1(0) = \theta_{p1}^0, \;\; \dot{\theta}_2(0) = \theta_{p2}^0. \tag{21}$$

By noting

$$\theta_1 = z_1, \; \theta_2 = z_2, \; \dot{\theta}_1 = z_3, \; \dot{\theta}_2 = z_4 \;, \tag{22}$$

the equations (19) become

$$\begin{cases} \dot{z}_1 = z_3, \\ \dot{z}_2 = z_4, \\ \dot{z}_3 = -w\sin z_1 - \gamma(z_1,z_2)\alpha[\cos z_1 + \xi\sin(z_2-z_1)], \\ \dot{z}_4 = -\beta w\sin z_2 + \gamma(z_1,z_2)\tilde{\alpha}[\cos z_2 + \xi\sin(z_2-z_1)], \end{cases} \tag{23}$$

with the initial conditions (21)

$$z_1(0) = z_1^0, \;\; z_2(0) = z_2^0, \;\; z_3(0) = z_3^0, \;\; z_4(0) = z_4^0. \tag{24}$$

We now consider that $|z_p| \le \frac{\pi}{2}$ and let approximate the trigonometric functions by series expansions [24]

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \varepsilon(z), \quad |\varepsilon(z)| \le 2\cdot 10^{-4},$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \varepsilon(z), \quad |\varepsilon(z)| \le 9\cdot 10^{-4}. \tag{25}$$

The system of equations (23) can be written as

$$\begin{cases} \dot{z}_1 = z_3, \\ \dot{z}_2 = z_4, \\ \dot{z}_3 = -wP(z_1) - \alpha Q_1(z_1,z_2)\gamma(z_1,z_2), \\ \dot{z}_4 = -\beta wP(z_2) + \tilde{\alpha} Q_2(z_1,z_2)\gamma(z_1,z_2), \end{cases} \tag{26}$$

with

$$\begin{aligned} & P(z) = z + \tilde{a}z^3 + \tilde{b}z^5,\ Q_1(z_1,z_2) = R_1(z_1) + R_2(z_1,z_2), \\ & Q_2(z_1,z_2) = R_1(z_2) + R_2(z_1,z_2),\ R_1(z) = 1 + \tilde{c}z^2 + \tilde{d}z^4, \\ & \gamma(z_1,z_2) = -1 + f^{-1/2}(z_1,z_2), \\ & R_2(z_1,z_2) = -\xi z_1 + \xi z_2 + \tilde{a}\xi z_2^3 - 3\tilde{a}\xi z_2^2 z_1 + \\ & \quad + 3\tilde{a}\xi z_2 z_1^2 - \tilde{a}\xi z_1^3 + \tilde{b}\xi z_2^5 - 5\tilde{b}\xi z_2^4 z_1 + \\ & \quad + 10\tilde{b}\xi z_2^3 z_1^2 - 10\tilde{b}\xi z_2^2 z_1^3 + 5\tilde{b}\xi z_2 z_1^4 - \tilde{b}\xi z_1^5, \\ & f\left(z_1,z_2\right) = 1 - 2\xi z_1 + 2\xi z_2 + 4\tilde{c}\xi^2 z_1 z_2 - 2\tilde{c}\xi^2 z_1^2 - \\ & -2\tilde{c}\xi^2 z_2^2 - 2\tilde{a}\xi z_1^3 + 2\tilde{a}\xi z_2^3 - 2\tilde{d}\xi^2 z_2^4 - 2\tilde{d}\xi^2 z_1^4 + 8\tilde{d}\xi^2 z_2^3 z_1 - \\ & -12\tilde{d}\xi^2 z_2^2 z_1^2 + 8\tilde{d}\xi^2 z_1^3 z_2 + 2\tilde{b}\xi z_2^5 - 2\tilde{b}\xi z_1^5, \end{aligned} \tag{27}$$

with $\tilde{a} = -\dfrac{1}{3!},\ \tilde{b} = \dfrac{1}{5!},\ \tilde{c} = -\dfrac{1}{2!},\ \tilde{d} = \dfrac{1}{4!}$.

Next, we consider a particular case for that the equation of motion admits an analytically solution represented by a sum of a linear superposition of cnoidal vibrations and a nonlinear superposition of cnoidal vibrations [29].

We consider the uncoupled case $\left(\alpha = \tilde{\alpha}\right)$

$$\begin{cases} \ddot{\theta}_1 + w\sin\theta_1 + \alpha\cos\theta_1 = 0, \\ \ddot{\theta}_2 + \beta w\sin\theta_2 - \alpha\cos\theta_2 = 0 \end{cases} \tag{28}$$

and associated initial conditions

$$\theta_i\left(0\right) = \theta_{i0},\ \dot{\theta}_i\left(0\right) = \theta_{ip0},\ i - 1,2. \tag{29}$$

Multiplying the first equation by $2\dot{\theta}_1$, and the second one by $2\dot{\theta}_2$, and integrating we obtain

$$\begin{aligned} & \dot{\theta}_1^2 = 2w\cos\theta_1 - 2\alpha\sin\theta_1 + C_1, \\ & \dot{\theta}_2^2 = 2w\beta\sin\theta_2 + 2\alpha\sin\theta_2 + C_2, \end{aligned} \tag{30}$$

with $C_i, i=1,2$ integration constants. Approximating the trigonometric functions by series of five-order we have

$$\dot{\theta}_i^2 = P_i(\theta_i),\ i=1,2, \tag{31}$$

where $P_i(\theta_i)$ are polynomials of fifth-order in $\theta_i$

$$P_i(\theta_i) = a_{0i} + a_{1i}\theta_i + a_{2i}\theta_i^2 + a_{3i}\theta_i^3 + a_{4i}\theta_i^4 + a_{5i}\theta_i^5,\ i=1,2, \tag{32}$$

with

$$\begin{aligned}
&a_{01} = 2w + C_1,\ a_{02} = 2\beta w + C_2,\\
&a_{11} = -2\alpha,\ aA_{12} = 2\alpha,\\
&a_{21} = 2w\tilde{c},\ a_{22} = 2\beta w\tilde{c},\\
&a_{31} = -2\alpha\tilde{a},\ a_{32} = 2\alpha\tilde{a},\\
&a_{41} = 2w\tilde{d},\ a_{42} = 2w\tilde{d}\beta,\\
&a_{51} = -2\alpha\tilde{b},\ a_{52} = 2\alpha\tilde{b},
\end{aligned}$$

where, for sake of simplicity, we take $-2w = C_1,\ -2\beta w = C_2$ and $a_{11} = -a_{12} = 2\alpha \neq 0$. The equations (3.4) are similar with the Weierstrass equation [33]

$$\dot{\theta}^2 = A_1\theta + A_2\theta^2 + A_3\theta^3 + A_4\theta^4 + A_5\theta^5, \tag{33}$$

where

$$A_1 = \frac{1}{2}a_1,\ A_2 = a_2,\ A_3 = \frac{3}{2}a_3,\ A_4 = 2a_4,\ A_5 = \frac{5}{2}a_5.$$

We know that the equation (33) admits a particular solution expressed as an elliptic Weierstrass function that is reduced, in this case, to the cnoidal function $cn$ [20].

$$\theta \equiv \wp(t; g_1, g_2) = e_2 - (e_2 - e_3)\,cn^2\left(\sqrt{e_1 - e_3}\,t; m\right), \tag{34}$$

where $e_1, e_2, e_3$ are the real roots of equation $4y^3 - g_1 y - g_2 = 0$ with $e_1 > e_2 > e_3$ and $g_1, g_2 \in R$ expressed in terms of the constants $A_i, i = 1,2,...,5$, and satisfying the condition $g_1^3 - 27g_2^2 > 0$. The modulus $m$ of the Jacobean elliptic function is $m = \dfrac{e_2 - e_3}{e_1 - e_3}$.

For arbitrary initial conditions (29)

$$\theta(0)=\theta_0,\dot{\theta}(0)=\theta_{p0}, \tag{35}$$

the solution of (33) can be written as a linear superposition of cnoidal vibrations of the form (34)

$$\theta_{lin}=2\sum_{k=0}^{n}\alpha_k cn^2\left[\omega_k t;m_k\right], \tag{36}$$

where the moduli $0\le m_k\le 1$, the frequencies $\omega_k$ and the amplitudes $\alpha_k$ depend on $\theta_0,\theta_{p0}$ and $A_k$.

To find the expression of the nonlinear superposition of cnoidal vibrations, we adopt the solution under the form

$$\theta=\frac{\lambda\wp(t)}{1+\mu\wp(t)}, \tag{37}$$

where $\wp(t)$ is the Weierstrass elliptic function given by (37), and $\lambda$ and $\mu$ arbitrary constants. Substituting (37) into (33) we obtain four equations for $\lambda,\mu,g_1$ and $g_2$. By solving this system of equation we have

$$\lambda=-30\left(15\alpha^2\tilde{b}\right)^{-3/2},\ \alpha^2\tilde{b}>0 \tag{38}$$

$$\mu=30\left(\frac{5\tilde{b}}{3}\right)^{1/4}\left(15\alpha^2\tilde{b}\right)^{-3/2}. \tag{39}$$

The constants $g_1$ and $g_2$ are obtained from the equations

$$6\lambda+\frac{3}{2}\lambda\mu^2 g_1=6A_1\mu^2+3A_2\lambda\mu+A_3\lambda^2, \tag{40}$$

$$\lambda\mu g_1+2\lambda\mu^2 g_2=A_1\mu+A_2\lambda. \tag{41}$$

So, a particular solution of (33) is written as a nonlinear superposition of cnoidal vibrations of the form (37), i.e.

$$\theta_{nonlin}=\frac{\lambda\left[e_2-(e_2-e_3)cn^2\left(\sqrt{e_1-e_3t}\right)\right]}{1+\mu\left[e_2-(e_2-e_3)cn^2\left(\sqrt{e_1-e_3t}\right)\right]}. \tag{42}$$

For arbitrary initial conditions (35) the nonlinear part of the solution of (33) can be written as

$$\theta_{nonlin} = \frac{\sum_{k=0}^{n} \beta_k cn^2\left[\omega_k t; m_k\right]}{1 + \sum_{k=0}^{n} \gamma_k cn^2\left[\omega_k t; m_k\right]}, \tag{43}$$

where the moduli $0 \le m_k \le 1$, $\omega_k$, $\beta_k$ and $\gamma_k$ depend on $\theta_0$, $\theta_{p0}$ and $A_k$ .

So, we can conclude that the solution of the equation (33) and the arbitrary initial conditions (35) is given by

$$\theta = \theta_{lin} + \theta_{nonlin}, \tag{44}$$

with $\theta_{lin}$ and $\theta_{nonlin}$ given by (36) and (43).

Next, we see that the theta-function is used for deriving the multiphase solutions of equations that can be reduced to Weierstrass equations with polynomials of higher order.
Let us introduce the following dependent-variable transformation [20, 26]

$$\theta = 2\frac{d^2}{dt^2}\log\Theta_n(t). \tag{45}$$

Then, the equation (36) is rewritten in the form of a bilinear differential equation

$$D_t^2(1 + D_t^2)\Theta_n \cdot \Theta_n = 0, \tag{46}$$

where the operator $D_t$ is defined as

$$D_t^m a \cdot b = (\partial_t - \partial_{t'})^m a(t)b(t')|_{t=t'}. \tag{47}$$

The function $\Theta_n$ is given in terms of an asymptotic expansion of the type (near-identity)

$$\Theta_n = 1 + \varepsilon\Theta_n^{(1)} + \varepsilon^2\Theta_n^{(2)} + \ldots, \tag{48}$$

with

$$\Theta_n^{(1)} = \sum_{i=1}^{n} \exp(i\omega_i t). \tag{49}$$

For $n = 1$ we have

$$\begin{aligned}
&\Theta_1 = 1 + \exp(i\omega_1 t + B_{11}),\\
&\Theta_2 = 1 + \exp(i\omega_1 t + B_{11}) + \exp(i\omega_2 t + B_{22}) + \exp(\omega_1 + \omega_2 + B_{12}),\\
&\Theta_3 = 1 + \exp(i\omega_1 t + B_{11}) + \exp(i\omega_2 t + B_{22}) + \exp(i\omega_3 t + B_{33}) +\\
&+\exp(\omega_1 + \omega_2 + B_{12}) + \exp(\omega_1 + \omega_3 + B_{13}) + \exp(\omega_2 + \omega_3 + B_{23}) +\\
&+\exp(\omega_1 + \omega_2 + \omega_3 + B_{12} + B_{13} + B_{23}).
\end{aligned} \tag{50}$$

For an arbitrary $n$ we have

$$\Theta_n = \sum_{M=0,1} \exp(i\sum_{i=1}^{n} M_i \omega_i t + \frac{1}{2}\sum_{i<j}^{n} B_{ij} M_i M_j), \tag{51}$$

where

$$\exp B_{ij} = \left(\frac{\omega_i - \omega_j}{\omega_i + \omega_j}\right)^2, \ \exp B_{ii} = \omega_i^2, \tag{52}$$

where $M = [M_1, M_2]$ is the vector of integer indices (0 and 1), $\omega = [\omega_1, \omega_2, ...\omega_n]$ the frequency vector, and $n$ is the finite number of degrees of freedom for a particular solution. This matrix $B$ is written as a sum of a diagonal matrix and an off-diagonal matrix $B = D + O$. The solution (45) can be written under the form [35]

$$\theta = 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\Theta(t) = 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log G(t) + 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left[1 + \frac{F(t)}{G(t)}\right], \tag{53}$$

where

$$G(t) = \sum_M \exp\left(iM\omega t + \frac{1}{2}M^T DM\right), \tag{54}$$

$$F(t) = \sum_M \left[\exp M^T OM - 1\right]\exp\left(iM\omega t + \frac{1}{2}M^T DM\right). \tag{55}$$

Consequently, the general solution (53) may be written in the terms of the *theta function* representation [20]

$$\theta(x,t) = \frac{2}{\lambda}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\Theta_n(\eta_1, \eta_2, ..., \eta_n), \tag{56}$$

where $\lambda = \alpha / 6\beta$ and $\Theta$ is the *theta function* defined as

$$\Theta_n\left(\eta_1,\eta_2,...,\eta_n\right) = \sum_{M\in(-\infty,\infty)} \exp\left( i\sum_{i=1}^{n} M_i\eta_i + \frac{1}{2}\sum_{i,j=1}^{n} M_i B_{ij} M_j \right), \tag{57}$$

with $n$ the number of degrees of freedom for a particular solution of the equation, and

$$\eta_j = k_j x - \omega_j t + \phi_j,\ 1 \le j \le N. \tag{58}$$

Here, $k_j$ are the wave numbers, the $\omega_j$ are the frequencies and the $\phi_j$ are the phases. Let us introduce the vectors of wave numbers, frequencies and constant phases

$$\begin{gathered} k = \left[k_1,k_2,...,k_n\right],\quad \omega = \left[\omega_1,\omega_2,...,\omega_n\right], \\ \phi = \left[\phi_1,\phi_2,...,\phi_n\right],\quad \eta = \left[\eta_1,\eta_2,...,\eta_n\right]. \end{gathered} \tag{59}$$

The vector $\eta$ can be written as

$$\eta = kx - \omega t + \phi. \tag{60}$$

Also, we can write

$$\begin{aligned} &M\eta = Kx - \Omega t + \Phi, \\ &M = \left[M_1, M_2, ..., M_n\right], \\ &K = Mk,\ \Omega = M\omega,\ \Phi = M\phi. \end{aligned}$$

The integer components in $M$ are the integer indice and the matrix $B$ can be decomposed in a diagonal matrix $D$ and an off-diagonal matrix $O$, that is

$$B = D + O. \tag{61}$$

Let us take

$$\begin{aligned} F_i = &\sum_{p=1}^{n} a_{ip}\theta_p + \sum_{p,q=1}^{n} b_{ipq}\theta_p\theta_q + \sum_{p,q,r=1}^{n} c_{ipqr}\theta_p\theta_q\theta_r + \\ &+ \sum_{p,q,r,l=1}^{n} d_{ipqrl}\theta_p\theta_q\theta_r\theta_l + \sum_{p,q,r,l,m=1}^{n} e_{ipqrlm}\theta_p\theta_q\theta_r\theta_l\theta_m + ..., \end{aligned} \tag{62}$$

with $i = 1,2,...,n$, and $a,b,c,...$ constants, the system of equations has the remarkable property that it can be reduced to Weierstrass equations,e the function transformation

$$\theta = 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\Theta_n(t), \tag{63}$$

with $\Theta_n(t)$ defined as (50) and

$$\Theta_n = \sum_{M\in(-\infty,\infty)} \exp\left( i\sum_{i=1}^{n} M_i\omega_i t + \frac{1}{2}\sum_{i<j}^{n} B_{ij} M_i M_j \right), \tag{64}$$

$$\exp B_{ij} = \left( \frac{\omega_i - \omega_j}{\omega_i + \omega_j} \right)^2, \quad \exp B_{ij} = \omega_i^2 . \tag{65}$$

In this way, the solution (63) becomes

$$\theta(t) = 2\frac{\partial^2}{\partial t^2}\log\Theta_n(\eta) = \theta_{lin}(\eta) + \theta_{\mathrm{int}}(\eta), \tag{66}$$

for $\eta = -\omega t + \phi$. The first term $\theta_{lin}$ represents a linear superposition of cnoidal waves, because we recognize in

$$\theta_{lin} = \sum_{l=1}^{n} \alpha_l \left[ \frac{2\pi}{K_l\sqrt{m_l}} \sum_{k=0}^{\infty} \left[ \frac{q_l^{k+1/2}}{1+q_l^{2k+1}} \cos(2k+1)\frac{\pi\omega_l t}{2K_l} \right]^2 \right] \tag{67}$$

the expression [20]

$$\theta_{lin} = \sum_{l=1}^{n} \alpha_l \mathrm{cn}^2[\omega_l t; m_l], \tag{68}$$

with

$$q = \exp\left(-\pi\frac{K'}{K}\right), \quad K = K(m) + \int_0^{\pi/2} \frac{\mathrm{d}u}{\sqrt{1 - m\sin^2 u}},$$

$$K'(m_1) = K(m), \quad m + m_1 = 1.$$

The second term $\theta_{\mathrm{int}}$ represents a nonlinear superposition or interaction among cnoidal waves

$$2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left[1 + \frac{F(t)}{G(t)}\right] \approx \frac{\beta_k \mathrm{cn}^2(\omega t, m_k)}{1 + \gamma_k \mathrm{cn}^2(\omega t, m_k)} . \tag{69}$$

If $m_k$ take the values 0 or 1, the relation (69) is directly verified. For $0 \le m_k \le 1$, the relation is numerically verified with an error of $|e| \le 5 \times 10^{-7}$. Consequently,

$$\theta_{nonlin} = \frac{\sum_{k=0}^{n} \beta_k \mathrm{cn}^2[\eta, m_k]}{1 + \sum_{k=0}^{n} \lambda_k \mathrm{cn}^2[\eta, m_k]}. \tag{70}$$

As a result, the cnoidal method yields to solutions consisting of a linear superposition and a nonlinear superposition of cnoidal waves [20].

By differentiating (69) with respect to time we obtain a relation with Tzitzeica surface [18, 19]

$$\begin{aligned} &ah = h_u,\, a_v = ba'' + h,\, b_v + bb'' = 0, \\ &b''h = h_v,\, a'' + aa'' = 0,\, b''_u + a''b = h, \end{aligned} \tag{71}$$

where $a, a', a''...$ are the centro-affine invariant functions that defines the position vector $r(u,v)$ of a surface $\Sigma$ in $\mathrm{R}^3$

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k,\ (u,v) \in D(t), \tag{72}$$

where $D(t)$ is a family of vector fields generated by $\{\psi_t\} = \{u(t), v(t)\}$.
We verified that the vector $r(u,v)$ satisfies the condition

$$(r, r_u, r_v) \ne 0, \tag{73}$$

and this is the reason that it can be considered the solution of the second-order partial differential equations system that defines a surface leaving a centro-affinity aside

$$r_{uu} = ar_u + br_v + cr,\ r_{uv} = a'r_u + b'r_v + c'r,\ r_{vv} = a''r_u + b''r_v + c''r, \tag{74}$$

$$(r_{uu})_v = (r_{uv})_u,\ (r_{uv})_v = (r_{vv})_u, \tag{75}$$

where $a, a', a''...$ are the centroaffine invariant functions of $u$ and $v$. For $c = c'' = 0$, the surface $\Sigma$ is related to the asymptotic lines.

## 3. THE OPTIMIZATION PROBLEM

The optimization problem considers the geometrical characteristics of the robot represented in. Fig. 2. We consider a 6 DOF parallel robot with 6 rotational, universal spherical and open kinematic loops. The active joint is the rotational one.

The center of the mobile platform and the origin of the relative coordinate system are represented together with the coordinates of the position vector $p$ of the point $P$ with respect to the absolute frame $O_0X_0Y_0Z_0$ are $\left(p_x, p_y, p_z\right)$.

The $\rho_x, \rho_y, \rho_z$ are the rotation angles of the system $PXYZ$ around $X_0$, $Y_0$ and $Z_0$. The points $S_i$ with the position vectors $s_i\left(s_{ix}, s_{iy}, s_{iz}\right)$ represent the spherical joints, and $U_i$ are points with the position vectors $u_i\left(u_{ix}, u_{iy}, u_{iz}\right)$ representing the universal joints. The points $R_i$ with the position vectors $r_i\left(r_{ix}, r_{iy}, r_{iz}\right)$ represent the base rotational joints. The radii of the base and upper circles are $R$ and $r$. The length the of the first segment of the kinematic open loop is $l_1$. The length of the second segment of the kinematic loop is $l_2$.The angle between $O_0X_0$ and $O_0R_i$ is $\beta_i$ .The angle between $PX$ and $PS_i$ is $\alpha_i$.

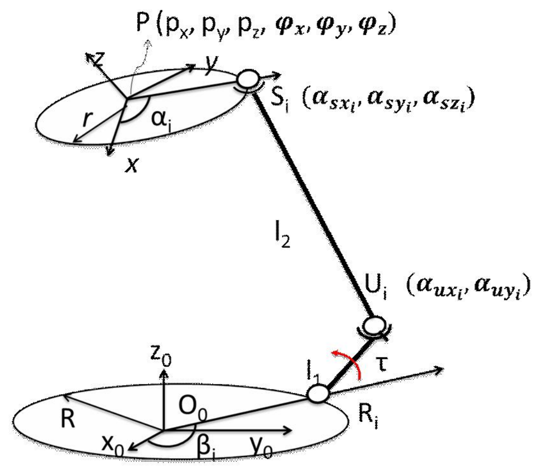


Fig. 2 – The robot.

We chose the design vector $x = \left[l_1, l_2, \text{ratio}_d, \text{ratio}_u, R, r\right]$m where the variable $\text{ratio}_d$ represents the ratio between two consecutive angles $\beta_i$ and $\beta_{i+1}$, $i = 1,3,5$, and $\text{ratio}_u$ is the ratio between two consecutive angles $\alpha_i$ and $\alpha_{i+1}$, where $i = 1,3,5$.

In the Fig. 3 a representation of the fixed and mobile platform is illustrated and the angles $\beta_i$ and $\alpha_i$ where $i = 1\ldots6$. The rotational joints and the spherical joints are symmetrically positioned around the center of the fixed and mobile platform respectively. The values of the parameters $\text{ratio}_d$ and $\text{ratio}_u$ are

$$\text{ratio}_d = \frac{\beta_1}{\beta_2} = \frac{\beta_3}{\beta_4} = \frac{\beta_5}{\beta_6}, \tag{76}$$

$$\text{ratio}_u = \frac{\alpha_1}{\alpha_2} = \frac{\alpha_3}{\alpha_4} = \frac{\alpha_5}{\alpha_6}. \tag{77}$$

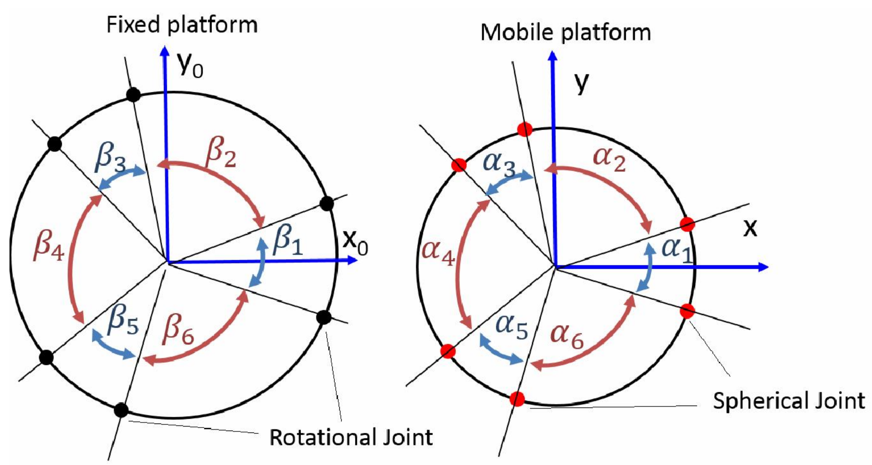


Fig. 3 – Fixed and the mobile platform.

We optimize the structure of the 6 DOF parallel robot for a prescribed workspace. The workspace to the robot is a parallelepiped of known dimensions. The optimization problem is to find $x$ that minimizes $f = f(x)$, subjected to the constraints $g_h(x) \le 0$, for $h = 0,1,2...,m$ and $k_h(x) = 0$, for $h = 0,1,2,...,p$, and the objective function

$$\text{objective function} = \frac{1}{n}\sum_{i=1}^{n} A_i C = \frac{1}{n}\sum_{i=1}^{n}\sqrt{(a_i c)^2 + (a_i' c')^2 + (a_i' c'')^2}\ . \tag{78}$$

Here, $x$ is the vector of the variables of the robot design (length of elements, geometrical characteristics, and the value of the torques available in the active joints), $f$ is the objective function that has to be minimized (in this case the function that evaluates the correlation of the workspace and the forces and torques). The function $f$ is also called fitness or cost function, $g_j(l)$ and $k_j(l)=0$ are the inequality and equality constraints

In order to evaluate the optimization function, it is considered that the imposed workspace is not included in the workspace of the parallel robots. In this case, the relationship between the distances $\text{all}(Dist_{min}) \le Dist_{Point\_WS}$ is not respected anymore. Therefore, the objective function returns the infinite value $fit = +\infty$. The constraints of the optimization define the minimum or maximum values of the variables from the design vector. The constraints imposed to the variables are

$$l_1 > 0.001\,\text{m}\ ,\ l_2 > 0.001\,\text{m},$$

$$-l_1 - l_2 - 1.1(r+R) < 0,\ -l_1 + l_2 - 1.1(r+R) < 0,$$

$$-l_1 + l_2 - 1.1(r+R) < 0,\ l_1 - l_2 < 0\,\text{m},$$

$$\text{ratio}_d > 0.5\ ,\ \ \text{ratio}_u > 0.5,$$

$$r > 0.001\ \text{m}\ ,\ R > 0.001\,\text{m}\ .$$

There were no constraints related to the maximal values for the length of the stroke arm and the linkage ($l_1$ and $l_2$) and to the radiuses of the mobile and fixed platform ( $r$ and $R$). The constraints have to be transformed in a matric equation in order to be inserted into the optimization algorithm.

The objective function of this optimization evaluates the correlation between an imposed parallelepiped and the workspace of the parallel robot. Therefore, an optimal structure of the parallel robot leads to a workspace that includes completely parallelepiped and the remaining workspace has the volume as low as possible. Since both of the workspaces are given as a scattered point cloud data, the relation between the workspaces is given by the relation between all the points from both of them.

## 4. RESULTS

The optimization problem leads to results represented in the Figs. 4–7. The vibrations $\theta_1(t)$ are shown in Fig. 4 for

$$\theta_1(0) = 1.5,\ \theta_2(0) = 0.5,\ \dot{\theta}_1(0) = 0.05,\ \dot{\theta}_2(0) = 0.05. \tag{79}$$

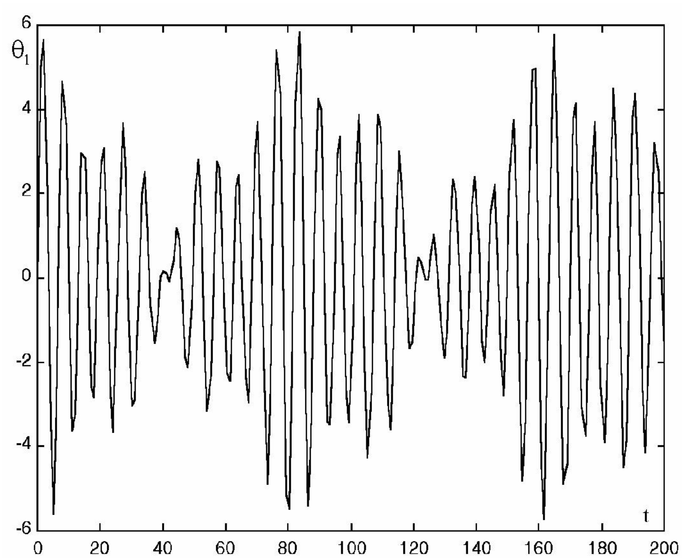


Fig. 4 – The vibration $\theta_1(t)$.

For

$$\theta_1(0) = \theta_2(0) = 0.045,\ \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0.02 \tag{80}$$

the vibration are displayed in Figs. 6 and 7.
The appearance of the vibrations, although they seem arbitrary, tends towards to a periodic arrangement.

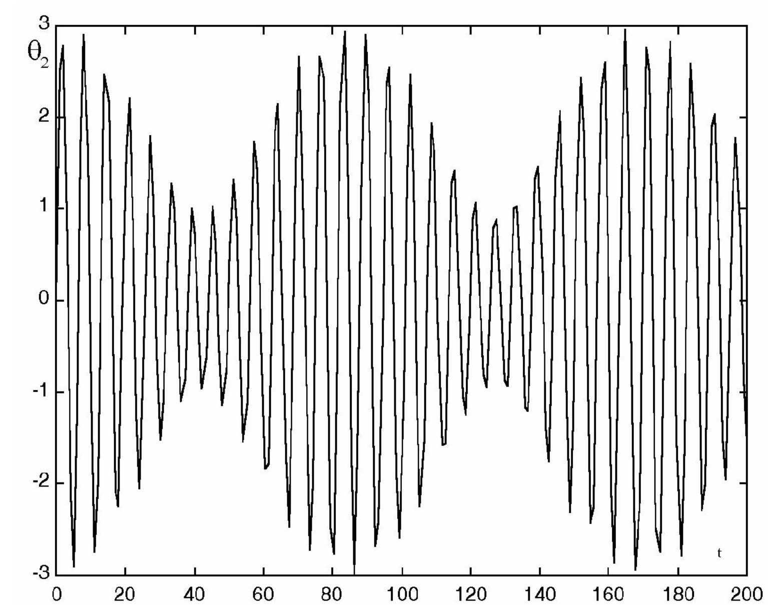


Fig. 5 – The vibration $\theta_2(t)$.

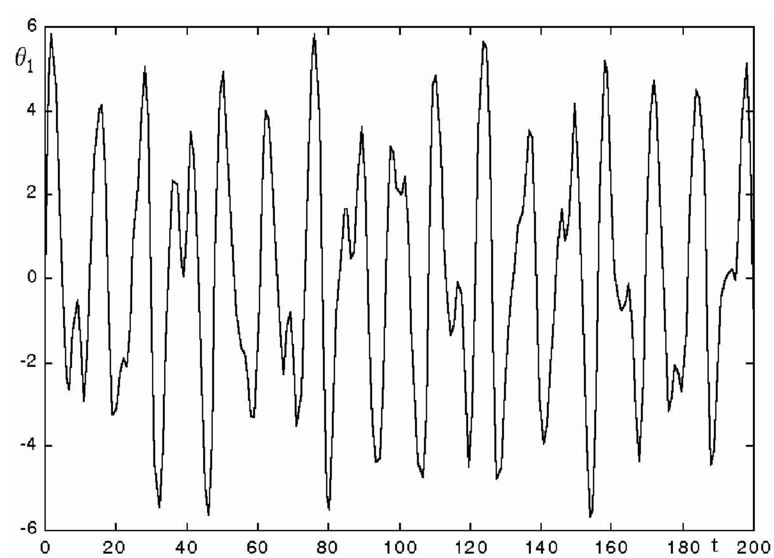


Fig. 6 – The vibration $\theta_1(t)$.

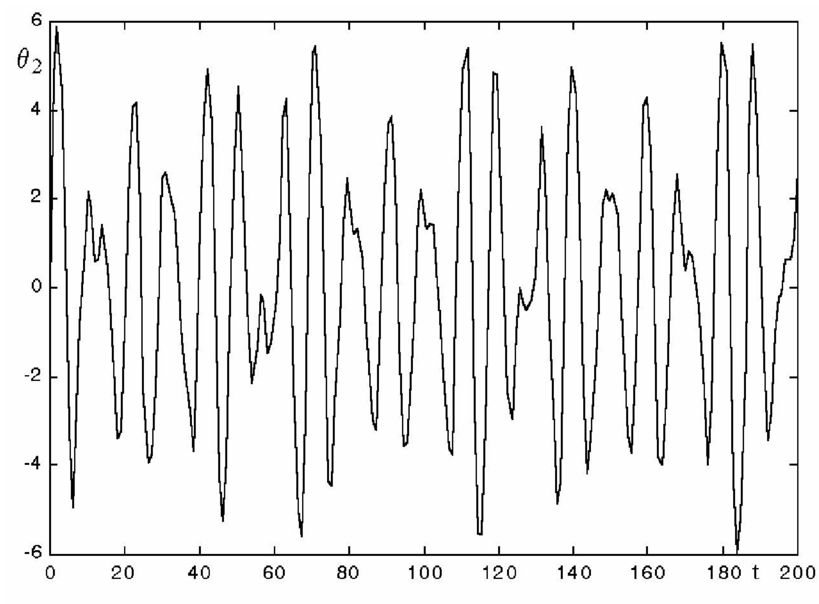


Fig. 7 – The vibration $\theta_2(t)$.

## 5. CONCLUSIONS

In this paper we analyse the equations that describe the motion of a system composed from a coupled pendula. The optimization problem considers a 6 DOF parallel robot with 6 rotational, universal spherical and open kinematic loops. The active joint is the rotational one. The cnoidal theory and a genetic algorithm are used to solve the problem. In all cases we have assumed that the number of populations is 25, ratio of reproduction is 1, number of multi-point crossovers is 1, probability of mutation is 0.2 and maximum number of generations is 250. The final parameters were obtained after 149 iterations.

**Acknowledgements**. This work was supported by a grant of the Romanian Ministry of Research and Innovation, project PN-IIIP2-2.1-PED-2019-0085 CONTRACT 447PED/2020 (Acronim POSEIDON).

*Received on November15, 2022*

## REFERENCES


1. ABLOWITZ, M. J., SEGUR, H., *Solitons and the inverse scattering transforms*, SIAM, Philadelphia, 1981.
2. ABLOWITZ, M. J., RAMANI, A., SEGUR, H., *A connection between nonlinear evolution equations and ordinary differential equations of P-type I, II*, J. Math. Phys., **21**, *4*, pp. 715–721, pp. 1006–1015, 1980.
3. ABRAMOWITZ, M., STEGUN, I. A. (eds.), *Handbook of mathematical functions*, U. S. Dept. of Commerce, 1984.
4. ACHENBACH, J. D., *Wave propagation in elastic solids*, North-Holland Publishing Company, Amsterdam, 1973.
5. ADAM, D., KOPF, F., *Theoretical Analysis of Dynamically Loaded Soils*, European Workshop:Compaction of Soils and Granular Materials; ETC11 of ISSMGE: Paris, 2000.
6. BAIRD, P., DANIELO, L., *Three-dimensional Ricci solitons which project to surfaces*, J .fur die reine and engewandte Mathematik Walter de Gruyter, Berlin, pp. 65–91, 2007.
7. CHEN, B.-Y., *A survey on Ricci solitons on Riemannian submanifolds*, Contemporary Mathematics, **674**, 2016.
8. BALA, N., *Symmetry groups and Lagrangians associated to Tzitzeica surfaces*, Report DG/9910138, 1999.
9. BLASCHKE, W., *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie, II*, Affine Differentialgeometrie, Grundlehren der math. Wissenschaften 7, Springer, Berlin, 1923.
10. BRATU, P., STUPARU, A., POPA, S., IACOB, N., VOICU, O., *The assessment of the dynamic response toseismic excitation for constructions equipped with base isolation systems according to the NewtonVoigt-Kelvin model*, Acta Tech. Napoc. Ser. Appl. Math. Eng., **60**, *4*, pp. 459–464, 2017.
11. BRATU, P., STUPARU, A., POPA, S., IACOB, N.. VOICU, O., IACOB, N., SPANU, G., *The dynamic isolation performances analysis of the vibrating equipment with elastic links to a fixed base*, Acta Tech.Napoc. Ser. Appl. Math. Eng., **61**, *1*, pp. 23–28, 2018.
12. CHIROIU, C., DELSANTO, P. P., SCALERANDI, M., CHIROIU, V., SIRETEANU, T., *Subharmonic generation in piezoelectrics with Cantor-like structure*, J. of Physics D: Applied Physics, **34**, *3*, pp. 1579–1586, 2001.
13. CHIROIU, C., MUNTEANU, L., CHIROIU, V., DELSANTO, P. P., SCALERANDI, M., *A genetic algorithm for determining the elastic constants of a monoclinic crystal,* Inverse Problems, Inst. Phys. Publ., **16**, *1*, pp. 121–132, 2000.
14. CHIROIU, V., CHIROIU, C., *Inverse Problems in Mechanics*, Publishing House of the Romanian Academy, Bucharest, 2003.
15. ROGERS, C., SCHIEF, W. K., *Bäcklund and Darboux transformations. Geometry and modern applications in soliton theory*, Cambridge Univ. Press, 2002.
16. ROGERS, C., SCHIEF, W. K., *The classical Bäcklund transformation and integrable discretisation of characteristic equations*, Phys. Letters A, *232*, pp. 217–223, 1997.
17. TODA, M. WADATI, M., *A soliton and two solitons in an exponential lattice and related equations*, J.of the Phys. Soc. of Japan, **34**, pp. 18–25, 1973.
18. TZITZEICA, G., *Sur une nouvelle classe des surfaces*, C. R. Acad. Sci., 1910.
19. TZITZEICA, G., *Géométrie projective différentielle des réseaux*, Editura Cultura Nationala, Bucharest and Gauthier-Villars, Paris, 1924.
20. CHIROIU, V., CHIROIU, C., BADEA, T., RUFFINO, E., *A nonlinear system with essential energy influx: the human cardiovascular system*, Proc. of Romanian Acad., Series A: Mathematics, Physics, Technical Sciences, Information Science, **1**, *1*, pp. 41–45, 2000.

21. CHIROIU, V., CHIROIU, C., RUGINA, C., DELSANTO, P.P., SCALERANDI, M., *Propagation of ultrasonic waves in nonlinear multilayered media*, J. of Comp. Acoustics, **9**, *4*, pp. 1633–1646, 2001.
22. CHIROIU, V., MUNTEANU, L., CHIROIU, C., DONESCU, St., *The motion of a micropolar fluid in inclined open channels*, Proc. of the Romanian Acad., Series A: Mathematics, Physics, Technical Sciences, Information Science, **4**, *2*, pp. 129–135, 2003.
23. CRASMAREANU, M., *Cylindrical Tzitzeica curves imply forced harmonic oscillators*, Balkan J. of Geometry and its Applications, **7**, pp. 37–42, 2002.
24. DOBRESCU, C. F., BRAGUTA, E., *Optimization of Vibro-Compaction Technological Process Considering Rheological Properties*, In Springer Proceedings in Physics, Proceedings of the 14th AVMS Conference, Timisoara, Springer, **198**, pp. 287–293, 2017.
25. DOBRESCU, C. F., *Highlighting the Change of the Dynamic Response to Discrete Variation of Soil Stiffness in the Process of Dynamic Compaction with Roller Compactors Based on Linear Rheological Modeling*, Appl. Mech. Mater., **801**, pp. 242–248, 2015.
26. DODD, R. K., EILBECK, J. C., GIBBON, J. D., MORRIS, H. C., *Solitons and Nonlinear Wave Equations*, Academic Press, London, NewYork, 1982.
27. LEOPA, A., DEBELEAC, C., NASTAC, S., *Simulation of Vibration E ects on Ground Produced by Technological Equipments*, In Proceedings of the 12th International Multidisciplinary Scientific GeoConference and EXPO-Modern Management of Mine Producing, Geology and Environmental Protection, SGEM 2012, Albena, Bulgaria, **5**, pp. 743–750, 2012.
28. MUNTEANU, L., CHIROIU, V., BRATU, P., *On the Ricci soliton equation-Part I. Cnoidal Method*, Romanian Journal of Mechanics, **5**, *2*, pp. pp. 22–42, 2020.
29. MUNTEANU, L., DONESCU, St., *Introduction to Soliton Theory: Applications to Mechanics*, Book Series Fundamental Theories of Physics, 143, Kluwer Academic Publishers, Dordrecht, Boston, 2004.
30. MUSETTE, M., CONTE, R., VERHOEVEN, C., *Bäcklund transformation and nonlinear superposition formula of the Kaup-Kupershmidt and Tzitzeica equations*, Centre de Recherches Mathematiques CRM Proc. and Lecture Notes, **29**, pp. 345–362, 2001.
31. NAYFEH, A., *Perturbation methods*, Wiley, NY, 1973.
32. NETTEL, S., *Wave Physics- Oscillations- Solitons- Chaos*, Springer-Verlag, 1992.
33. NEWELL, A. C., *Solitons in mathematics and physics*, SIAM, Philadelphia, 1985.
34. OLVER, P. J., *Evolution equations possessing infinite symmetries*, J. of Math. Phys., **18**, *6*, pp. 1212–1215, 1977.
35. OSBORNE, A. R., *Soliton physics and the periodic inverse scattering transform*, Physica D, **86**, pp. 81–89, 1995.
36. PAINLEVE, P., *Leçons sur la théorie analytique des équations differentielles. Leçons de Stockholm*, Hermann, 1895, Paris 1897, Reprinted in Oeuvres de Paul Painlevé, vol. 1, editions du CNRS, Paris, 1973.