TAILORING OF FUNCTIONALLY GRADED SPHERES USING A UNIFORM STRESS CONDITION

ŞTEFAN SOROHAN , DAN MIHAI CONSTANTINESCU*

Abstract. Homogeneous and isotropic thick spheres loaded with constant internal and/or external pressures can not be economically designed because the maximum equivalent stress is a local value. It was analytically demonstrated that, by neglecting the body loads, a functionally graded material (FGM) may be characterized in linear static analyses by two material constants: Young's modulus $E(r)$ and Poisson's ratio $\nu(r)$. If these two functions are both known, the solution of the problem, displacement and stress distributions may be relatively easy obtained. For the inverse problem, in which a desired stress combination distribution is imposed, finding of $E(r)$ and $\nu(r)$ is more difficult, if such a solution exists. More than that, if the solution exists, it is not unique, because two unknown functions are involved. For $\nu(r)$ = const., analytical solutions are available for $E(r)$, but only for two particular stress conditions. In this paper, the inverse problem is solved iteratively using a finite element model and an algorithm of stress uniformization developed by the authors of this paper is proposed. In this original approach, the existing solutions were reproduced as a verification and afterwards new solutions were obtained for the remaining classical theories of resistance. The new obtained solutions were also verified by using the analytical solutions of the direct problem.

Keywords: thick spheres, elastic analysis, functional graded material (FGM), finite element analysis (FEA), full stressed design (FSD), iterative technique.

1. INTRODUCTION

The distribution of the stresses in a structure usually shows their maximum values in a limited number of points. If the topology is fixed, as in the case of pressurized spheres, the variation of the stresses can be modified by changing the material properties according to a constitutive law as to obtain the same maximum equivalent stress in a larger domain, preferably in the whole structure. This problem may be defined as an optimization one.

For an inhomogeneous isotropic material, locally defined through Young's modulus E and Poisson's ratio ν, obtained through the combination of at least two components, one "soft" and one "rigid", may result important variations of the Young's modulus, usually in the order of units and also some variations in the

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Poisson's ratio. Such materials, called functionally graded materials (FGMs) represent a class of novel materials in which compositions/constituents and/or microstructures gradually change along single or multiple spatial directions, resulting in a gradual change in properties and functions which can be tailored for enhanced performance. These materials can be obtained through two phases, for example metal and ceramic, whose volume fractions vary gradually, porous materials [1] and additive manufacturing technologies [2, 3].

Optimization per se, although proposed more than 100 years ago, evolved considerably. There is an extremely large number of information and publications in this domain, well-known books, as Haftka and Gürdal [4], but recently the used approaches and the mathematical apparatus have diversified considerably [5]. The classical analytical analyses were extended towards the numerical ones and today most of the practical applications are based on the finite element analysis (FEA). The dedicated software packages do not generally contain explicit implementation for modeling FGMs, as most of them use isotropic or anisotropic homogeneous materials. This drawback was initially overcomed by implementing finite elements with an inhomogeneous material, but a sufficiently good precision can be obtained by using refined meshes for which each element is considered an isotropic material [5, 6].

A large category of academic applications dedicated to the optimization issues were developed by using truss structures [7, 8]. These can be dimensioned as being fully stressed, that is in all bars made from the same material the optimum design is obtained when same stress is obtained, and the *fully stressed design* (FSD) concept is used. Recently the authors of this paper used a similar approach of FSD for designing truss and continuous plane structures using FGMs instead of changing the cross section areas [9].

For simple FGM structures loaded symmetrically as spheres and thick-walled tubes, the stress conditions (circumferential stress and in-plane shear stress) for which the whole geometrical domain reaches constant values were established, that is the FSD design is used and the necessary distribution $E(r)$ is obtained in the hypothesis of keeping $\nu(r)$ = const. [10, 11]. In real problems both $E(r)$ and $\nu(r)$ variables and not always known. For solving such relatively complex problems, assuming a particular power low or an exponential low variation for $E(r)$ and keeping again $\nu(r) = \text{const.}$, usually analytical closed form solutions in some particular load cases are obtained [12–15]. Numerical methods with complex approaches [16–20] can be used to obtain solutions for given laws of variation for the properties of the FGM. Closed-form solutions for a functionally graded circular hollow cylinder and radial expansion/contraction of a hollow sphere loaded by inner and outer pressures made of incompressible linear elastic materials, where $\nu(r)$ = const. = 0.5, were obtained by Batra [21, 22].

If a FGM is used for FSD, while Young's modulus is variable it is probable that the failure criteria and the density of the material will be variable, which leads this approach to a complicated problem. The method may become efficient from practical point of view if instead of designing under the fully stressed concept is used another parameter as the *structural efficiency percentage* [23] which is equivalent to attaining the limit state of stress in all points of a structure simultaneously.

Recently, functionally graded plates were analyzed by using the finite element method [24–26] bringing new insights into the development of engineering design.

This work is dedicated to tailoring the functionally graded spheres using a uniform stress condition. The stress condition may be one very simple as linear combinations of principal stresses or an equivalent stress. Starting from the fact that additive printing allows the obtaining of FGMs, it is desired to equalize some well-established variables. The present paper is a continuation of the research presented previously in [9].

First, in section two, the article briefly presents the existing analytically optimum design solution for FGM spheres. Then, in section three, the stress condition uniformization algorithm is briefly described in a finite element model implementation. In section four, all the stress conditions used for uniformizations are clearly presented. For verification, as pointing out the novelties of this paper, in section five two applications are given as examples. Comments on the obtained results are also presented. The paper ends with some general conclusions in section six.

2. ANALYTICAL SOLUTIONS

It is considered a sphere with inner radius a and outer radius b, loaded by an internal pressure p_i and external pressure p_o (Fig. 1), and having an elastic and isotropic material being either homogeneous or inhomogeneous (FGM). For small elastic strains in the spherical symmetric case, neglecting the body loads, the radial and circumferential stresses, σ_r and σ_t, must satisfy the equilibrium equation, as stated by Timoshenko and Goodier [27]

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}+\frac{2}{r}\left(\sigma_r-\sigma_t\right)=0\,, \tag{1}$$

and the corresponding strains, ε_r and ε_t, must satisfy the compatibility equations

$$\varepsilon_r=\frac{\mathrm{d}u}{\mathrm{d}r};\qquad \varepsilon_t=\frac{u}{r};\qquad \varepsilon_r=\frac{\mathrm{d}}{\mathrm{d}r}\left(r\varepsilon_t\right). \tag{2,a,b,c}$$

The stress-strain relations are:

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - 2\nu\sigma_t\right); \qquad \varepsilon_t = \frac{1}{E}\left[\sigma_t - \nu\left(\sigma_r + \sigma_t\right)\right]. \tag{3,a,b}$$

For a homogeneous material, the material properties E and ν are constant and results the stresses distribution given by [27] as

$$\begin{aligned}\sigma_r &= p_i \frac{a^3}{\left(b^3 - a^3\right)}\left(1 - \frac{b^3}{r^3}\right) - p_o \frac{b^3}{\left(b^3 - a^3\right)}\left(1 - \frac{a^3}{r^3}\right);\\ \sigma_t &= p_i \frac{a^3}{\left(b^3 - a^3\right)}\left(1 + \frac{1}{2}\frac{b^3}{r^3}\right) - p_o \frac{b^3}{\left(b^3 - a^3\right)}\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right).\end{aligned} \tag{4,a,b}$$

These stresses lead to important variations of the equivalent stresses $\sigma_{eqv}(r)$, calculated with any of the classical theories of resistance. For an isotropic and homogeneous material, the sum of the principal stresses (first invariant) is constant $\sigma_r + 2\sigma_t = \text{const}$.

Displacements are resulting from (2,b), that is $u = \varepsilon_t r$ with ε_t to be obtained from (3,b) by considering the stresses given by relations (4).

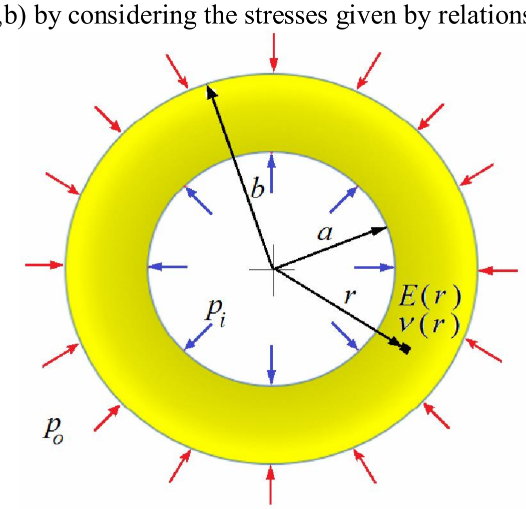

Fig. 1 – FGM sphere.

Considering $E(r)$ variable but $\nu(r) = \text{const.}$, are obtained solutions [10, 11] for which

$$\sigma_t - \sigma_r = \text{const.} \tag{5}$$

By considering the simplified calculation of the principal stresses in a sphere, as two of the principal stresses are always equal, it results that equation (5) corresponds also to the equivalent stresses calculated with the theories of maximum shear stress and maximum distortion strain energy.

Using (1) – (3), (5) and the boundary conditions expressed through (6)

$$\sigma_r(a) = -p_i; \qquad \sigma_r(b) = -p_o, \tag{6,a,b}$$

is obtained the expression for the variation of the modulus of elasticity which satisfies relation (5), that is [10, 11]

$$E(r) = C\left\{-p_i\left[1-\nu+2(1-2\nu)\log\frac{r}{b}\right]+p_o\left[1-\nu+2(1-2\nu)\log\frac{r}{a}\right]\right\}^{\frac{3(\nu-1)}{2(2\nu-1)}}, \tag{7}$$

in which C is a constant, and the relations of variation of the stresses are described by

$$\begin{aligned} \sigma_r(r) &= \frac{p_i \log\frac{b}{r} - p_o \log\frac{a}{r}}{\log\frac{a}{b}}; \\ \sigma_t(r) &= -\frac{p_i}{2\log\frac{a}{b}}\left(1-2\log\frac{b}{r}\right)+\frac{p_o}{2\log\frac{a}{b}}\left(1-2\log\frac{a}{r}\right). \end{aligned} \tag{8,a,b}$$

Similarly, the analytical solutions are obtained for one more case if it is imposed that

$$\sigma_t = \text{const.}, \tag{9}$$

relation which, for certain conditions, may correspond to the equivalent stress calculated as for the maximum stress theory, and for which it was obtained [10, 11]

$$E(r) = C\left\{p_i\frac{a}{b}\left[(1-2\nu)+\nu\frac{b^2}{r^2}\right]-p_o\frac{b}{a}\left[(1-2\nu)+\nu\frac{a^2}{r^2}\right]\right\}^{\frac{\nu-1}{2\nu}}, \tag{10}$$

with C being also a constant; the relations for the variation of the principal stresses are given by

$$\begin{aligned} \sigma_t(r) &= -p_i\frac{a^2}{a^2-b^2}+p_o\frac{b^2}{a^2-b^2}; \\ \sigma_r(r) &= -p_i\frac{a^2}{a^2-b^2}\left(1-\frac{b^2}{r^2}\right)+p_o\frac{b^2}{a^2-b^2}\left(1-\frac{a^2}{r^2}\right). \end{aligned} \tag{11,a,b}$$

For relations (7) and (10), it should be stated that $E(r) > 0$ for r in the interval $[a, b]$, therefore additional analyses for the existence of the solution are needed but they are not discussed in the present; however, it has to be mentioned that there are not always solutions to be found for any combination for the loading p_o/p_i, respectively any b/a.

As for the homogeneous material, the displacements for the FGM material are resulting from relation (2,b), with ε_t to be obtained from (3,b) by replacing the stresses from relations (8) or (11), and the longitudinal modulus of elasticity from relations (7) and (10), depending on the theory of resistance which is used.

It is to be noticed that relations (7) and (10) show that the solution, from the point of view of the distribution of the material, is not unique even if ν = const., therefore constant C can be established with one of the following relations

$$E(r=a)=E_a \quad \text{or} \quad E(r=b)=E_b, \tag{12,a,b}$$

with E_a and E_b being the moduli of elasticity at the interior and exterior of the sphere. Usually, in engineering applications one of these two conditions is initially imposed.

3. ALGORITHM FOR STRESS UNIFORMIZATION

Essentially, this optimization algorithm was presented by the authors in paper [9]. Shortly, we resume this algorithm to be applied for a FGM sphere. Due to the symmetry, given by geometry, loading and material distribution, an axial-symmetric model of the sphere for a portion of angle θ can be built, small enough as to reduce the size of the model (see Fig. 2).

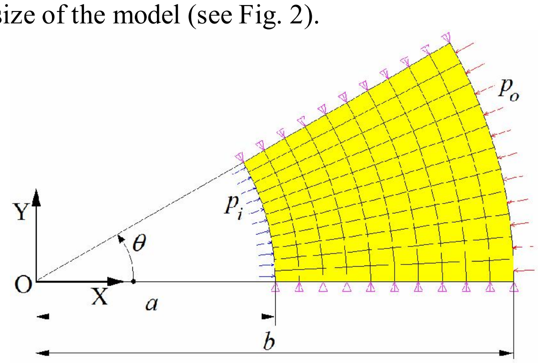

Fig. 2 – Finite element axi-symmetric model for a part of the sphere. For each finite element j is assigned a material j with Young's modulus E_j.

For the finite element model, meshed uniformly with a total of n Q4 or Q8 finite elements, the Young's moduli of all elements E_j, $j = 1\ldots n$, are considered as design variables. It is defined

$$\boldsymbol{X} = \begin{bmatrix} E_1 & E_2 & \cdots & E_n \end{bmatrix}, \tag{13}$$

the vector of unknowns or the design variables. For each finite element is associated an average value of a stress combination, for example an equivalent stress $\sigma_{eqv,j}$. A function $F : R^n \to R$ can be defined through the extreme global values of the equivalent stresses in the sphere $\sigma_{eqv,\max} = \max\limits_j\left(\sigma_{eqv,j}\right)$ and $\sigma_{eqv,\min} = \min\limits_j\left(\sigma_{eqv,j}\right)$. The following function is defined

$$F\left(\boldsymbol{X}\right) = \sigma_{eqv,\max} - \sigma_{eqv,\min}, \tag{14}$$

using the two extreme values of the stress definition. If the extreme values are equal, then $F\left(\boldsymbol{X}\right) = 0$ and the equivalent stresses are constant and uniformly distributed over the whole domain, i.e. being equivalent to the FSD condition. The problem of the optimization to be stated becomes a continuous variable optimization criterion

$$\mathit{Minimize} \quad F\left(\boldsymbol{X}\right), \tag{15}$$

as an objective function subjected to side constraints

$$0 < E_{\min} \le E_j \le E_{\max}; \quad j = 1\ldots n, \tag{16,a}$$

where $E_{\min}$ and $E_{\max}$ define the domain of acceptable values for the design variables. If necessary, a supplementary constraint

$$-1 < \nu_{\min} \le \nu_j\left(E_j\right) < \nu_{\max} = 0.5; \quad j = 1\ldots n, \tag{16,b}$$

may be considered, i.e. the coefficient of transversal contraction ν_j is not considered constant, but it is a function of E_j.

Using the adapted FSD algorithm, the relation of iterative “correction” of the modulus of elasticity was chosed as

$$E_j^{(i)} = E_j^{(i-1)}\left[1 + \alpha^{(i)}\left(1 - \frac{\sigma_{eqv,j}^{(i-1)}}{\sigma_{eqv,\max}^{(i-1)}}\right)\right], \tag{17}$$

where:

- $\alpha^{(i)}$ defines the “degree of modification” of the modulus of elasticity at current iteration (i) and which in this work was maintained constant $\alpha = \alpha^{(i)}$; it may be updated in every iteration to reduce the number of total iteration until convergence;

- $\sigma_{eqv,j}^{(i-1)}$ is the stress condition, which is to become uniform, as equivalent stress in element *j* at the iteration (*i*-1);

- $\sigma_{eqv,\max}^{(i-1)}$ is the maximum equivalent stress for all elements at iteration (*i*-1).

Initially, for $i = 1$, it is considered that all the elements *j* are made from the same material with the modulus of elasticity $E_j^0 = E_{\min}$, and for each new iteration (*i*+1), in each element the modulus of elasticity (as a design variable) is updated. It was checked that this condition may be also a random function as $E_j^0 = \text{rand}\,(E_{\min}, E_{\max})$, but the number of iterations until convergence increases. If the initialization of the initial E_j^0 is closer to the final distribution of $E(r)$, then the number of iterations until convergence decreases.

For stopping the calculus (given by the convergence criterion) the infinity norm of relative variation of the modulus of elasticity was used for the last two iterations,

$$\max_j \left(\frac{\left| E_j^{(i)} - E_j^{(i-1)} \right|}{E_j^{(i)}} \right) \le \varepsilon_E \,, \tag{18,a}$$

where it can be chosen that $\varepsilon_E = 10^{-3} - 10^{-6}$, that is in between several orders of magnitude.

The second condition for ending the calculation is based on the extreme stresses obtained at the current iteration, that is

$$\frac{\sigma_{eqv,\max}^{(i)} - \sigma_{eqv,\min}^{(i)}}{\sigma_{eqv,\max}^{(i)}} \le \varepsilon_\sigma \,, \tag{18,b}$$

where, again, ε_σ can be chosen in between $10^{-3} - 10^{-6}$. This criterion is efficient for problems of FSD type, whereas the first one is useful for problems where constraints (16) are to be imposed. In this paper the iteration procedure stops when one of them is first fulfilled.

As to avoid obtaining negative values of the modulus of elasticity and for obtaining a unique solution for the distribution $E(r)$ it was arbitrarily chosen for the minimum value of the modulus to be exactly E_0. Thus, if necessary, at each iteration the obtained current values $E_j^{(i)}$ can be uniformly scaled using

$$E_j^{(i)} = \frac{E_0}{\min_j \left(E_j^{(i)} \right)} E_j^{(i)} \,; \quad \text{for} \quad j = 1 \ldots n \,. \tag{19}$$

The algorithm was implemented in ANSYS APDL for the model presented in Fig. 2 using Plane182 element type with reduced integration, more details about implementation being given in [9].

4. PARTICULAR STRESS CONDITIONS

In this paper, different stress conditions are used for uniformization in equation (17). First, the classical theories of resistance are included and then a new arbitrarily simple stress condition was considered. The radial and circumferential stresses are sorted as principal stresses, $\sigma_1 \geq \sigma_2 \geq \sigma_3$. According to [28, p. 473] if we assume that the allowable stresses in tension and compression are the same in absolute value, then:

I. according to maximum stress theory

$$\sigma_{eqv}^{I} = \max\left(\left|\sigma_1\right|,\left|\sigma_3\right|\right); \tag{20}$$

with | | used for absolute value.

II. according to maximum strain theory

$$\sigma_{eqv}^{II} = \max\left(\left|\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right|,\left|\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right|\right); \tag{21}$$

III. according to maximum shear theory

$$\sigma_{eqv}^{III} = \sigma_1 - \sigma_3; \tag{22}$$

IV. according to maximum total strain energy theory

$$\sigma_{eqv}^{IV} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)}; \tag{23}$$

V. according to maximum distortion strain energy theory

$$\sigma_{eqv}^{V} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2}. \tag{24}$$

For a sphere, because two of three principal stresses are always equals, it results, $\sigma_{eqv}^{III} = \sigma_{eqv}^{V} = \left|\sigma_t - \sigma_r\right|$. So, for a sphere, the design condition, according to (5), is the same in obtaining constant equivalent stresses as given by (22) and (24).

If $\left|\sigma_t\right| \geq \left|\sigma_r\right|$, then according to (20) it results $\sigma_{eqv}^{I} = \left|\sigma_t\right|$ and the design condition (9) is the same as given by (20).

A very simple stress design condition (chosen arbitrarily) is also analyzed in this paper, as considering the sum of the extremes of principal stresses

$$\sigma_{cond} = \sigma_1 + \sigma_3. \tag{25}$$

5. APPLICATIONS

The results obtained with finite element computations for the uniformization of a combination of stresses can be describes piecewise (as considered constant for each finite element) for the variation of the modulus of elasticity. Through approximate interpolation (fitting) with a function proposed in [15] as

$$E(r) = E_a \exp\left(\log \frac{E_b}{E_a} \frac{1-\left(\frac{r}{a}\right)^{\eta}}{1-\left(\frac{b}{a}\right)^{\eta}} \right), \tag{26}$$

with η a constant, and it results an analytical description of the modulus of elasticity sufficiently well approximated. Relation (26) can be used for obtaining an analytical solution by considering the approach given by Nejad et al. [15] and can verify the results given through FEA. It was established that the solution obtained in closed form by Tutuncu and Ozuturk [12] and later corrected by Shy and Xie [13] in which the description of the variation of $E(r)$ is given through a power-law function cannot approximate very well the numerical results given by FEA and therefore the verification of the results obtained by uniformization cannot be done perfectly.

For the general case, when $E(r)$ and $\nu(r)$ are given as arbitrarily functions, there are no closed form analytical solutions. The approximate numerical solution based on a Fredholm integral equation was proposed by Li et al. [17], paper in which was suggested that in practice it is better to use an exponential form for the $E(r)$ variation, in a similar form as suggested by Nejad et al. [15].

5.1. APPLICATION 1 – SPHERE ONLY WITH INTERNAL PRESSURE

As a first case study we consider a sphere with a = 75 mm; b = 100 mm; p_i = 100 MPa and p_o = 0. We look for the distribution of $E(r)$, if $\nu(r)$ = 0.3 = const., as to uniformize the equivalent stresses presented in relations (20) – (24) and the supplementary condition (25). As to obtain a unique solution is chosen arbitrarily $E_{\min}$ = 100 GPa. As from mathematical point of view the problem is unidimensional (1D) and the choice of angle θ is not important, we use the axisymmetric model (Fig. 2) with θ = 0.25º. The model was uniformly meshed in 500 elements on radial direction and 2 elements on circumferential direction, and 1000 elements resulted in total, with as many variables E_j. For the convergence criterion we considered $\varepsilon_E = \varepsilon_\sigma = 10^{-6}$. By considering the parameter $\alpha = 1.0$ or $\alpha = 0.5$ (only for the stress condition (25) as it resulted a lack of convergence because of the oscillations of the consecutive solutions for $\alpha = 1.0$), the algorithm of stress condition uniformization converged relatively rapidly.

By considering the maximum distortion strain energy theory, respectively the stress condition (24), three particular curves giving the convergence of Young's moduli and the two extremes of von Mises stresses are presented in Fig. 3 and Fig. 4. Radial and circumferential stress distributions obtained using the method presented in this paper and analytical solution (8) for the same case are presented in Fig. 5. The solutions obtained for the variation of $E(r)$ by considering all analyzed stress conditions for uniformization are presented in Fig. 6.

The maximum equivalent stresses obtained using the proposed method for all stress conditions discussed in this paper are presented in Table 1.

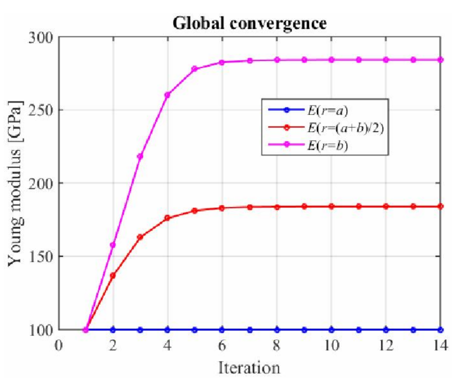

Fig. 3 – Young's modulus evolution in three locations by using the stress condition $\sigma_{eqv}^{V} = \text{const}$.

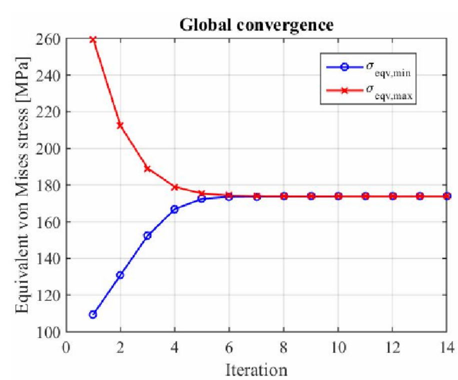

Fig. 4 – Extreme von Mises stresses variation for $\sigma_{eqv}^{V} = \text{const}$.

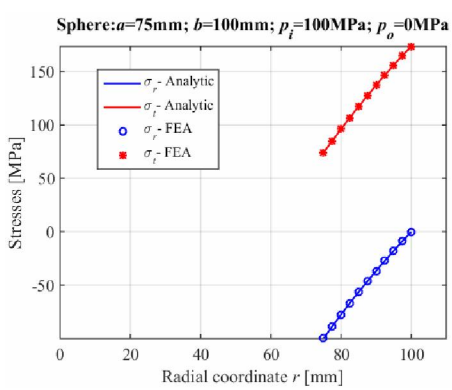

Fig. 5 – Stress component distribution in pressurized sphere for stress condition $\sigma_{eqv}^{V} = \text{const}$.

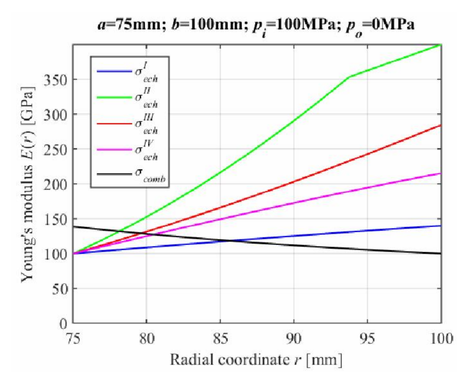

Fig. 6 – Young's modulus distributions $E(r)$ obtained using all stress conditions in this paper.

Regardless the theory of resistance used for stresses uniformization, their values (marked as bold) are inferior to the maximum values considered as reference which result for a homogeneous material. As it was expected, the use of an arbitrary chosen stress condition given by (25) i.e. last row in Table 1, does not lead to the optimization in the FGM design.

Table 1

Maximum equivalent stresses for different stress uniformization conditions - Application 1

Stress condition design	$\sigma^{I}_{eqv,\max}$	$\sigma^{II}_{eqv,\max}$	$\sigma^{III}_{eqv,\max}$	$\sigma^{IV}_{eqv,\max}$
Homogeneous, Eqs. (4) - reference	159.5	195.7	259.5	254.4
$\sigma^{I}_{eqv} = \lvert\sigma_t\rvert = \text{const.}$	**128.6**	177.0	228.4	220.3
$\sigma^{II}_{eqv} = \text{const.}$	181.8	**127.3**	195.9	216.2
$\sigma^{III}_{eqv} = \sigma^{V}_{eqv} = \lvert\sigma_t - \sigma_r\rvert = \text{const.}$	173.8	144.2	**173.8**	205.6
$\sigma^{IV}_{eqv} = \text{const.}$	152.0	154.3	190.6	**179.8**
$\sigma_{cond} = \sigma_1 + \sigma_3 = \text{const.}$	192.4	215.2	292.2	291.2

The results obtained numerically for $E(r)$ and interpolated as in (26) are presented in Tabel 2, and can be introduced directly as input data in the relations given in papers like [15] and [17] as to verify the solutions obtained here.

Table 2

Parameters in Eq. (26) which best approximate the numerical solutions of $E(r)$ with $\nu(r) = 0.3$ for Application 1

Stress condition design	E_a [GPa]	E_b [GPa]	η [-]	No. of total iterations	$\alpha = \text{const.}$ [-]
$\sigma^{I}_{eqv} = \lvert\sigma_t\rvert = \text{const.}$	100	139.8	-0.997	10	1.0
$\sigma^{II}_{eqv} = \text{const.}$	100	399.6	-3.81*	24	1.0
$\sigma^{III}_{eqv} = \sigma^{V}_{eqv} = \lvert\sigma_t - \sigma_r\rvert = \text{const.}$	100	284.2	-1.37	14	1.0
$\sigma^{IV}_{eqv} = \text{const.}$	100	215.1	-2.44	11	1.0
$\sigma_{cond} = \sigma_1 + \sigma_3 = \text{const.}$	138.5	100	-0.747	16	0.5

* Approximate fitting value as the FEA solution, continuous non-uniform (see Fig. 6) cannot be correctly described by (26)

5.2. APPLICATION 2 – SPHERE ONLY WITH EXTERNAL PRESSURE

For a sphere with a = 25 mm; b = 100 mm; p_i = 0 and p_o = -10 MPa it is to be found the distribution of $E(r)$, if $\nu(r)$ = 0.4 = const., in the same conditions as for the previous application. We choose $E_{\min}$ = 1.0 GPa and $\theta = 0.25^o$. Same model as before was considered. Now, for the convergence criterion we considered $\varepsilon_E = \varepsilon_\sigma = 10^{-4}$ and it was observed that for $\alpha = 0.6$ all the analyzed cases ensure convergence.

For stress condition (24), similar curves of convergence as in the previous application are presented in Fig. 7 and Fig. 8. Radial and circumferential stress distribution obtained using the FEA and analytical solution for the same case are presented in Fig. 9. The solution $E(r)$ for all analyzed constant stress conditions are presented in Fig. 10.

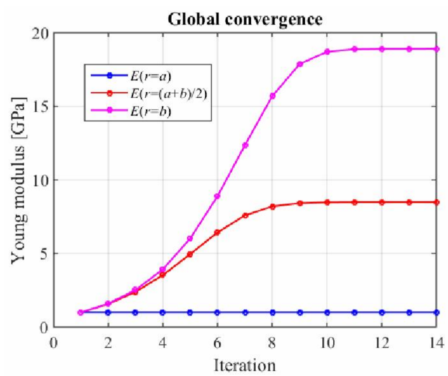

Fig. 7 – Young's modulus evolution in three elements - stress condition $\sigma^V_{eqv} = \text{const}$.

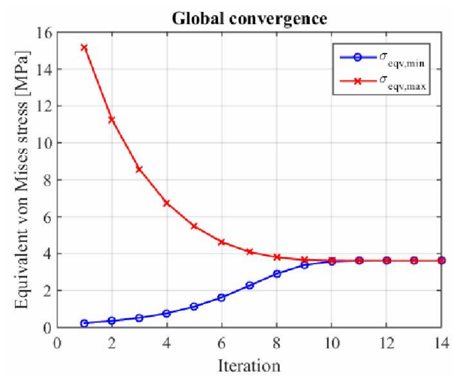

Fig. 8 – Extreme von Mises stresses evolution - stress condition $\sigma^V_{eqv} = \text{const}$.

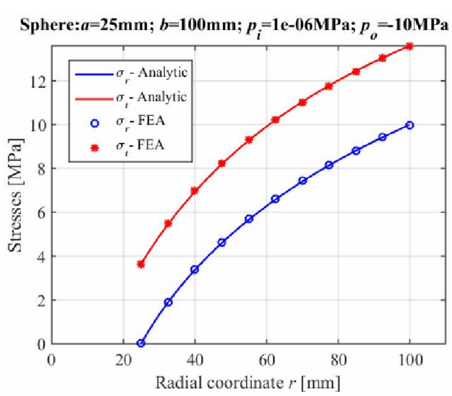

Fig. 9 – Stress component distribution for stress condition $\sigma^V_{eqv} = \text{const}$.

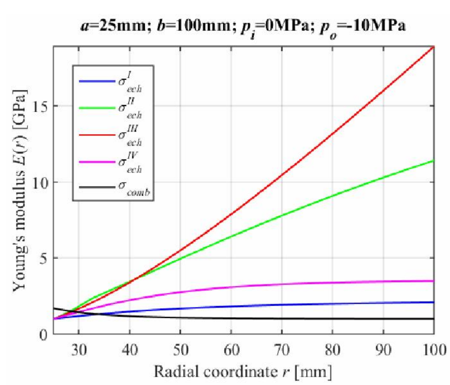

Fig. 10 – Young's modulus distributions $E(r)$ obtained using all stress condition.

Maximum equivalent stresses obtained using the proposed method for all stress conditions discussed in this paper are presented in Table 3. Regardless the theory of resistance used for the stress uniformization the obtained values (with bold in Table 3) are lower than the maximum values resulting for a homogeneous material considered as reference. As expected, by using an arbitrary stress condition (last row in Table 3), the optimization in the design of a sphere is not obtained.

Table. 3

Maximum equivalent stresses for different stress uniformization conditions - Application 2 with $\nu(r)$=0.4

Stress condition design	$\sigma^{I}_{eqv,\max}$	$\sigma^{II}_{eqv,\max}$	$\sigma^{III}_{eqv,\max}$	$\sigma^{IV}_{eqv,\max}$
Homogeneous, Eqs. (4) - reference	15.24	12.19	15.24	16.69
$\sigma^{I}_{eqv} = \lvert\sigma_t\rvert = \text{const.}$	**10.67**	8.501	10.63	11.66
$\sigma^{II}_{eqv} = \text{const.}$	12.28	**3.367**	4.811	9.192
$\sigma^{III}_{eqv} = \sigma^{V}_{eqv} = \lvert\sigma_t - \sigma_r\rvert = \text{const.}$	13.60	4.163	**3.607**	10.22
$\sigma^{IV}_{eqv} = \text{const.}$	10.91	5.951	7.444	**8.162**
$\sigma_{cond} = \sigma_1 + \sigma_3 = \text{const.}$	20.02	15.95	19.96	21.88

As before, the results obtained numerically for $E(r)$ and fitted with (26), which are presented in Tabel 4, can be introduced directly in the relations mentioned in papers [15] and [17] as to verify the solutions obtained by us. The obtained results for this second application in some stress condition design consist in very large ratio of extreme Young's moduli; for example, the maximum reported value E_b / E_a = 18.9, for the stress condition (5) is not a realistic value for current FGM, but for the same problem in stress condition (9), the ratio reduces to only 2.08. This is way, in almost all published papers, for large ratios b / a, the authors [11, 17] prefer to present results only for stress condition (9).

Table. 4

Parameters in Eq. (26) which best approximate the numerical solutions of $E(r)$ and $\nu(r)$=0.4 - Application 2

Stress condition design	E_a [GPa]	E_b [GPa]	η [-]	No. of total iterations	$\alpha = \text{const.}$ [-]
$\sigma^{I}_{eqv} = \lvert\sigma_t\rvert = \text{const.}$	1.00	2.08	-1.33*	31	0.6
$\sigma^{II}_{eqv} = \text{const.}$	1.00	11.4	-0.878*	21	0.6
$\sigma^{III}_{eqv} = \sigma^{V}_{eqv} = \lvert\sigma_t - \sigma_r\rvert = \text{const.}$	1.00	18.9	-0.451	14	0.6
$\sigma^{IV}_{eqv} = \text{const.}$	1.00	3.49	-2.32*	36	0.6
$\sigma_{cond} = \sigma_1 + \sigma_3 = \text{const.}$	1.70	1.00	-2.80*	31	0.6

* Approximate fitting values as the FEA solution cannot be correctly described by (26)

6. CONCLUSIONS

From the two applications presented it results that the proposed algorithm is efficient and leads to numerical solutions for problems without restrictions which have also an analytical solution. Besides the applications presented in this paper several other applications were analyzed and the proposed algorithm was validated.

For the situations in which restrictions are encountered in the domain of variation of the modulus of elasticity the results will lead to a considerable reduction of the equivalent stresses. The objective function which was presented in this paper was defined by relation (14), following the von Mises theory. For a sphere it is possible to obtain stress uniformization for all five classical theories of resistance. It is also possible to propose the optimization of other quantities instead of the equivalent stress, as the specific strain energy. Or, as trying to uniformize a coefficient of safety which is related to the modulus of elasticity and the yielding stress, then a relation between these two values, if know, can be considered an uniformization objective.

The present researches will continue as to fundament the mathematical aspects and in order to identify new practical applications in which the algorithm can lead to the optimization of the different material distributions. The proposed methodology can be extended to more complex practical problems in which, for example, in the stress combinations function the unknowns $E(r)$ and/or $\nu(r)$ are included.

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