NONLINEAR FORCED VIBRATION OF A NANOBEAM RESTING ON WINKLER-PASTERNAK ELASTIC FOUNDATION AND SUBJECTED TO A MECHANICAL IMPACT

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Abstract. The nonlinear governing equations of nanobeam taking into consideration its curvature, resting on an elastic Winkler-Pasternak foundation and based on non-local Euler-Bernoulli beam theory is analyzed. The equation of motion and the boundary conditions are modeled within the framework of a simple supported nanobeam which accounts the presence of a mechanical impact and nonlinear von-Karman strain. The resulting nonlinear differential equations are reduced to only one differential equation which is studied by means of the Optimal Auxiliary Functions Method (OAFM). An explicit analytical solution is proposed for a complex problem. The main quality of our technique consists in the existence of some auxiliary functions derived from the expressions of the solution for the initial linear equation and the form of nonlinear term calculated from the above solution of the linear equation. The convergence-control parameters present in the auxiliary functions are evaluated by a rigorous mathematical procedure. The obtained solutions are in very good agreement with the numerical solution.

Key words: nonlinear forced vibration, Optimal Auxiliary Functions Method, Winkler-Pasternak foundation, mechanical impact.

1. INTRODUCTION

The flexural vibration of a nanobeam under mechanical impact and resting on a Winkler-Pasternak foundation is a topic of interest for researchers because many structures include nanobeams: textile fibers, airplane wing, flexible satellite, paper sheets, band saw blades, robotic manipulators, oil pipelines, and so on. Nanobeams and microbeams have attracted considerable attention in the literature. For example, Zhao *et al*. [1] investigated the nonlinear equation of a microbeam considering the strain gradient and Hamilton's principle. The nonlinear static bending deformation,

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the post-buckling problem and nonlinear free vibration are analyzed in detail. Ghayesh [2] proposed the study of the forced non-linear vibrations of an axially moving beam fitted an intra-span spring-support which is solved by the pseudo arclength continuation technique. Bifurcation diagrams of Poincare maps are presented as either the forcing amplitude or axial speed is varied. The thermo-mechanical nonlinear vibration and stability of a hinged-hinged axially moving beam additionally supported by a nonlinear spring-mass support are examined by numerical procedures by Kazemirad [3]. The steady state resonant response of the system with and without an internal resonance between the first two modes are examined.

A nonclassical model for the electrostatically actuated microbeam is developed by Kong [4] using a modified couple stress theory and minimum potential energy principle. The approximate analytical solutions to the pull-in voltage and pull-in displacement of the microbeam are derived using Rayleigh-Ritz method and the static pull-in behavior of an electrostatically actuated microbeam is analyzed. A novel method is proposed by Rokni *et al*. [5] to obtain static pull-in voltage with fringing field effects in electrostatically actuated cantilever and clamped-clamped microbeams. The non-classical Euler-Bernoulli beam model is adopted to effectively capture the size effect and the governing fourth-order differential equation of variable coefficients is converted into a Fredholm integral equation. Peng *et al*. [6] explored the nonlinear electro-dynamic response of a microbeam under a step applied voltage, with a particular focus on the influence of the material nonlinearity. The governing equation is solved using Lindstedt-Poincare perturbation method combined with Galerkin method. Poorjamshidian *et al*. [7] proposed a geometric type of nonlinearity which is due to the stretching effect of a mid-plane of the beam. Analytical solution is obtained through a combination of the homotopy method and traditional perturbation procedure.

The non-local Euler-Bernoulli beam theory is employed by Togun and Bagdatli [8] in the nonlinear free and forced vibration analysis of a nanobeam resting on an elastic foundation of the Pasternak type. Forcing and damping effects are considered and the multiple scale method, a perturbation technique is applied to obtain the approximate solution. The Differential transformation method is utilized by Ebrahimi *et al*. [9] for vibration and buckling analysis of nanotubes in thermal environment while considering the coupled surface and non-local effects. The Eringen’s nonlocal elasticity theory considered the effect of small size while the Gurtin-Murdoch model is used to incorporate the surface effects. Tao *et al*. [10] examined the nonlinear dynamic behavior of fiber metal laminated beams subjected to moving loads in thermal environment. Galerkin and Newmark methods are used to obtain the dynamic responses of fiber-metal laminated beam. Pi and Ouyang [11] combined the positive position feedback control of beams with the sliding mode control to suppress the vibration of the beam when the moving mass is on and off the beam.

Zarepour *et al*. [12] presented electro-thermal-mechanical nonlinear vibration of nanobeam resting on the Winkler-Pasternak foundation. The nonlinear frequency for simply supported beam is obtained by means of differential transform method in conjunction with a program. The pull-in instability of a cantilever nano-actuator model incorporating the effect of the surface, the fringing field and the Casimir attraction force is investigated by Duan *et al*. [13]. A quadratic polynomial is proposed as the shape function for the cantilever beam, and the analytic expression for the tip deflection and pull-in parameters of the beam are derived using the Gaussian quadrature rule. Peng *et al*. [14] presented a nonlinear electro-dynamic analysis for a size dependent microbeam made of materials with nonlinear elasticity by employing the modified couple stress theory. Pradiptya and Ouakad [15] analyzed the nonlinear size dependent behavior of electrically actuated carbon nanotube-based nano-actuator while including the higher-order strain gradient deformation, the geometric nonlinearity, the slack effect, and the temperature gradient effects. A Jacobian method is applied to determine the variation of the frequencies of the nanobeam with the DC load.

Free vibration behavior of carbon nanotube-reinforced composite microbeams is considered by Civalek *et al*. [16]. Carbon nanotubes are distributed in a polymeric matrix with four cases of reinforcement. The obtained vibration equation is saved by means of Navier's method. Anh and Hieu [17] studied the effect of magnetic field on the nonlinear vibration response of the single walled carbon nanotubes based on nonlocal strain gradient theory. Using the equivalent linearization method with weighted averaging value, the expressions of the nonlinear frequencies are obtained in the analytical forms. Yinusa and Sobamowo explored nonlinear internal flow induced vibration and stability of a pre-tensioned nanotube that rests on elastic foundation [18]. The impact of nonlinear Winkler foundation is considered and incorporated, while the nonlinear differential equation is solved via classical differential method. The obtained analytical solution is further treated by the application of SAT and CAT after-treatment techniques to enables it captures large time domain during dynamic response.

The present study is devoted to nonlinear forced vibration of a nanobeam resting on a nonlinear Winkler-Pasternak elastic foundation, based on nonlocal Euler-Bernoulli theory, and subjected to a mechanical impact. The nonlinearity of the equations is caused by the curvature of the nanobeam and by elastic foundation. The governing equation is discretized by means of Galerkin-Bubnov procedure. The obtained nonlinear differential equation is solved using OAFM. A very accurate solution is obtained using a moderate number of convergence-control parameters via auxiliary functions.

2. THE GOVERNING EQUATION

Based on the Eringen nonlocal continuum theory, the stress at a reference point x in a body is considered as a function of the strains of all points from its neighborhood. This effect can be justified by the atomic theory of lattice dynamics and phonon dispersion's research [9].

For a homogeneous and isotropic elastic solid, the nonlocal stress tensor components σ_{ij} at any point x in the body can be expressed as:

$$\sigma_{ij}(x)=\int_{(V)} K(|x'-x|,\tau)t_{ij}(x')\mathrm{d}V(x'), \tag{1}$$

where t_{ij} are the components of the classical local stress tensor at point x, which are related with the components of the linear strain tensor ε_{ij} by the conventional constitutive relations for a Hookean material as:

$$t_{ij}=C_{ijkl}(x)E_{kl}(x), \tag{2}$$

where $C_{ijkl}(x)$ is the fourth-order elasticity tensor, $(x'-x)$ is the Euclidian distance, $K((x'-x),\tau)$ is the nonlocal kernel and τ is a constant given by:

$$\tau=\frac{e_0 a}{l}, \tag{3}$$

in which e_0 is constant appropriate to each material, a is internal characteristic length and l is external characteristic length (e.g. crack length, wavelength). The magnitude of e_0 is determined experimentally or approximated by matching the dispersion curves of plane waves with those of the atomic lattice dynamics.

In general, it is difficult to solve the elasticity problems by using the integral constitutive relation in Eq. (1), but it is possible to represent the integral constitutive relations in an equivalent differential form as:

$$(1-(e_0 a)^2\nabla^2)\sigma_{ij}(x)=t_{ij}(x), \tag{4}$$

where ∇ is the Laplacian operator. The constitutive equation of nonlocal elasticity for a beam takes the following form:

$$\sigma_{xx}-(e_0 a)^2\frac{\partial^2\sigma_{xx}}{\partial\sigma^2}=E\varepsilon_{xx}, \tag{5}$$

where σ and ε are the nonlocal stress and strain, respectively and E is the elasticity modulus.

This section is carried out based on the nonlocal Euler-Bernoulli beam model. This theory implies that the plane sections that are normal to the mid-plane of the beam remain straight and normal to the mid-plane after deformation, such that the effects of shear deformation and rotational inertia are not included in this case. The nanobeam is resting on a three parameters elastic foundation with the linear and nonlinear spring constants K_L and K_{NL} of the Winkler elastic medium and K_P spring constant of the Pasternak elastic medium (Fig. 1):

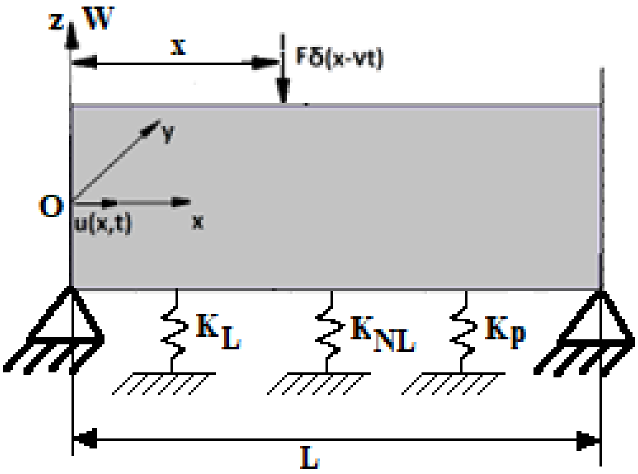

Fig. 1 – Modeling of simply-supported nanobeam with mechanical impact.

The Euler-Bernoulli beam theory proves that the displacement fields of any point are:

$$u_x(x,z,t) = u(x,t) - z\frac{\partial W(x,t)}{\partial x}; \quad u_y(x,z,t) = 0; \quad u_z(x,z,t) = W(x,t), \tag{6}$$

where u and W are the axial and transverse displacements of the point on the mid-plane of the beam. The axial strain ε_{xx} and shear strain γ_{xx} of the beam are (von Karman's nonlinear strain):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 - zk; \quad \gamma_{xz} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial x} = 0;$$
$$k = \frac{\partial^2 W}{\partial x^2}\Bigg/\left[1 + \left(\frac{\partial W}{\partial x}\right)^2\right]^{3/2}, \tag{7}$$

where k is the curvature of the beam.

The variational form of equation of motion can be obtained by Hamilton principle:

$$\delta\int_0^1 (K_e - U_s + W)\mathrm{d}t = 0, \tag{8}$$

where K_e is the kinetic energy, U_s is the strain energy and W is the work done by external applied forces.

The kinetic energy of the beam can be expressed as:

$$K_e = \frac{\rho A}{2}\int_0^1\left[\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right]dx. \tag{9}$$

The first variation of the strain energy can be calculated as:

$$\delta U_s = \int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V = \int_V (\sigma_{xx}\delta\varepsilon_{xx})\mathrm{d}V\ . \tag{10}$$

Substituting Eq. (7) into Eq. (10) one can get:

$$\delta U_s = \int_0^L\left[N\delta\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] - M\delta\left(\frac{\partial^2 W}{\partial x^2}\right)\right]\mathrm{d}x, \tag{11}$$

where N and M are the axial force and bending moment, respectively. These stress resultants used in Eq. (11) are defined as:

$$N = \int \sigma_{xx}\mathrm{d}A; \quad M = \int z\sigma_{xx}\mathrm{d}A, \tag{12}$$

where A is the area of the cross-section for the nanobeam.

The strain energy U_s can be written for the nanobeam as:

$$U_s = \frac{1}{2}\int_0^L\left\{N\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] - M\frac{\partial^2 W}{\partial x^2}\right\}\mathrm{d}x. \tag{13}$$

The virtual work by the external mechanical impulse and external load for the elastic medium of Winkler-Pasternak type is given by:

$$\delta W = \int_0^L q\delta W\mathrm{d}x, \tag{14}$$

where:

$$q = -F\delta(x-v) - (K_L W + K_{NL}W^3 - K_p\frac{\partial^2 W}{\partial x^2})\ . \tag{15}$$

Inserting Eq. (9), (13), (14) and (15) into Eq. (8) and integrating by parts, and collecting the coefficients of δ_u and δ_W, we obtain the following equations of motion:

$$\frac{\partial N}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2}\,; \tag{16}$$

$$\begin{aligned}&\frac{\partial^2 M}{\partial x^2} - \frac{\partial}{\partial x}\left(N\frac{\partial W}{\partial x}\right) + K_L W + K_{NL}W^3 - K_p\frac{\partial^2 W}{\partial x^2} + F\delta(x - vt) = \\ &= \rho A\frac{\partial^2 W}{\partial t^2} - \rho I\frac{\partial^4 W}{\partial t^2 \partial x^2}\end{aligned} \tag{17}$$

Integrating Eq. (5) over the cross-sectional area of the beam, we obtain the force-strain and the moment-strain relations of the nonlocal beam theory as follows:

$$N - (e_0 a)^2 \frac{\partial^2 N}{\partial x^2} = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right], \tag{18}$$

$$M - (e_0 a)^2 \frac{\partial^2 M}{\partial x^2} = -EI\frac{\partial^2 W}{\partial x^2}\Bigg/\left[1 + \left(\frac{\partial W}{\partial x}\right)^2\right]^{3/2}. \tag{19}$$

The explicit expressions of the nonlocal normal force and bending moment of the nanobeam can be derived by substituting Eq. (6) into Eq. (18) and Eq. (18) into Eq. (19), as follows:

$$N = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] + (e_0 a)^2 \frac{\partial^3 u}{\partial t^2 \partial x}, \tag{20}$$

$$\begin{aligned}M &= -EI\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 W}{\partial x^2}\Bigg/\left(1 + \left(\frac{\partial W}{\partial x}\right)^2\right)^{3/2}\right] + \\ &+ (e_0 a)^2\left[\rho A\frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial x}\left(N\frac{\partial W}{\partial x}\right) - \right. \\ &\left. - K_L W - K_{NL}W^3 + K_p\frac{\partial^2 W}{\partial x^2} - F\delta(x - vt) - \rho I\frac{\partial^4 W}{\partial t^2 \partial x^2}\right]\end{aligned} \tag{21}$$

Usually, the longitudinal inertial term $\partial^2 u/\partial t^2$ can be neglected, such that from Eq. (16) it is clear that N is a constant $N = C$, and therefore from (20) it holds that:

$$\frac{\partial u}{\partial x} = \frac{C}{EA} - \frac{1}{2}\left(\frac{\partial W}{\partial t}\right)^2 . \tag{22}$$

By integrating the last equation with the boundary conditions:

$$u(0, t) = u(L, t) = 0, \tag{23}$$

the constant C is given by:

$$C = N = -\frac{EA}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x . \tag{24}$$

According to Eq. (19), the term which defines the curvature of the beam can be written in the form:

$$\frac{\partial^2 W}{\partial x^2} \Bigg/ \left(1+\left(\frac{\partial W}{\partial x}\right)^2\right)^{3/2} \approx \frac{\partial^2 W}{\partial x^2}\left[1-\frac{3}{2}\left(\frac{\partial W}{\partial x}\right)^2\right], \tag{25}$$

such that the nonlinear vibration equation of motion for the nanobeam can be obtained by substituting Eqs. (21) and (25) into Eq. (19) as follows:

$$\begin{aligned}
&EI\left[\frac{\partial^4 W}{\partial x^4} - \frac{3}{2}\frac{\partial^4 W}{\partial x^4}\left(\frac{\partial W}{\partial x}\right)^2 - 3\left(\frac{\partial^2 W}{\partial x^2}\right)^3 - 9\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}\frac{\partial^3 W}{\partial x^3}\right] + \\
&+\rho A\frac{\partial^2}{\partial t^2}\left[W - (e_0 a)^2\frac{\partial^2 W}{\partial x^2}\right] + K_L\left[W - (e_0 a)^2\frac{\partial^2 W}{\partial x^2}\right] + \\
&+K_{NL}\left[W^3 - (e_0 a)^3\frac{\partial^3 W}{\partial x^3}\right] - K_P\frac{\partial^2}{\partial t^2}\left[W - (e_0 a)^2\frac{\partial^2 W}{\partial x^2}\right] + \\
&+\rho I\left[\frac{\partial^4 W}{\partial t^2 \partial x^2} - (e_0 a)^2\frac{\partial^6 W}{\partial t^2 \partial x^4}\right] - \frac{EA}{2L}\left[\frac{\partial^2 W}{\partial x^2} - \right. \\
&\left. -(e_0 a)^2\frac{\partial^2 W}{\partial x^2}\right]\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x = F\delta(x - vt).
\end{aligned} \tag{26}$$

The following non-dimensional quantities aiming to study the problem under general form are considered:

$$\bar{x}=\frac{x}{L},\quad \bar{W}=\frac{W}{L},\quad \bar{t}=\frac{t}{L^2}\sqrt{\frac{EI}{\rho A}},\quad \bar{\gamma}=\frac{e_0 a}{L},\quad \bar{K}_L=\frac{K_L L^4}{EI},$$
$$\bar{K}_{NL}=\frac{K_{NL}L^6}{EI},\bar{K}_P=\frac{K_P L^2}{EI},\quad \bar{v}=vL\sqrt{\frac{\rho A}{EI}},\quad \bar{f}=\frac{FL^4}{EI},\quad \bar{\alpha}=\frac{I}{AL^2}.$$

Omitting the bars and considering Eqs. (25) and (26), the non-dimensional form of the governing Eq. (25) can be expressed as:

$$\begin{aligned}&\frac{\partial^4 W}{\partial x^4}-\frac{3}{2}\frac{\partial^4 W}{\partial x^4}\left(\frac{\partial W}{\partial x}\right)^2-3\left(\frac{\partial^2 W}{\partial x^2}\right)^3-9\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}\frac{\partial^3 W}{\partial x^3}+\frac{\partial^2 W}{\partial t^2}\\&-\gamma^2\frac{\partial^4 W}{\partial x^2\partial t^2}+K_L\left(W-\gamma^2\frac{\partial^2 W}{\partial x^2}\right)+K_{NL}\Big[W^3-\\&-3\gamma^2\left(\left(\frac{\partial W}{\partial x}\right)^2 W+W^2\frac{\partial^2 W}{\partial x^2}\right)\Big]-\frac{1}{\alpha}\left(\frac{\partial^4 W}{\partial x^2\partial t^2}-\gamma^2\frac{\partial^6 W}{\partial t^2\partial x^4}\right)-\\&-\frac{\alpha}{2}\left(\frac{\partial^2 W}{\partial x^2}-\gamma^2\frac{\partial^4 W}{\partial x^4}\right)\int_0^1\left(\frac{\partial W}{\partial x}\right)^2\mathrm{d}x=f\delta(x-vt).\end{aligned}\tag{27}$$

The solution of Eq. (27) can be assumed to be of the form:

$$W(x,t)=X(x)T(t).\tag{28}$$

By substituting Eq. (28) into Eq. (27), multiplying Eq. (27) by $X(x)$ and integrating on the domain [0, 1], using the expression:

$$\int_0^1 f(x)\delta(x-vt)=f(vt),\tag{29}$$

one can obtain the following nonlinear differential equation:

$$\ddot{T}+\omega^2 T+aT^3=f_0(vt),\quad f_0(vt)=\frac{1}{\sqrt{2}}fX(vt),\tag{30}$$

where the dot denotes derivative with respect to time and the parameters which appear into Eq. (30) are:

$$\omega^2 = \frac{1}{q}\int_0^1\left[(1+K_P\gamma^2)\frac{\partial^4 X(x)}{\partial x^4}X(x) + K_L X^2(x) - \right.$$
$$\left. -(\gamma^2+K_P)\frac{\partial^2 X(x)}{\partial x^2}X(x)\right]\mathrm{d}x;$$
$$a = \frac{1}{q}\int_0^1\left[K_{NL}\left(X^4(x) - 6\gamma^2 X^2(x)\left(\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right)^2 - 3\gamma^2 X^3(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}\right) - \right.$$
$$-3X(x)\frac{\mathrm{d}^4 X(x)}{\mathrm{d}x^4}\left(\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right)^2 - 3X(x)\left(\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}\right)^3 -$$
$$\left. -9X(x)\frac{\mathrm{d}X(x)}{\mathrm{d}x}\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^3 X(x)}{\mathrm{d}x^3}\right]\mathrm{d}x - \frac{1}{2\alpha q}\left[\int_0^1\frac{\mathrm{d}X(x)}{\mathrm{d}x}\mathrm{d}x\right]\cdot \tag{31}$$
$$\cdot\left[\int_0^1\left(X(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} - \gamma^2 X(x)\frac{\mathrm{d}^4 X(x)}{\mathrm{d}x^4}\right)\right]\mathrm{d}x; \qquad f_0 = \frac{f}{q};$$
$$q = \int_0^1\left[X^2(x) - \gamma^2 X(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + \alpha\left(X(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} - \right.\right.$$
$$\left.\left. -\gamma^2 X(x)\frac{\mathrm{d}^6 X(x)}{\mathrm{d}x^6}\right)\right]\mathrm{d}x.$$

In the present study we consider the case of a simply supported beam, and therefore the boundary conditions are

$$W(0,t) = \frac{\partial^2 W(0,t)}{\partial x^2} = \frac{\partial^4 W(0,t)}{\partial x^4} = 0,$$
$$W(1,t) = \frac{\partial^2 W(1,t)}{\partial x^2} = \frac{\partial^4 W(1,t)}{\partial x^4} = 0. \tag{32}$$

The eigenfunction $X(x)$ from Eq. (28) can be expressed from Eq. (32) as

$$X(x) = \sqrt{2}\sin\pi x, \tag{33}$$

which satisfies the orthogonality condition

$$\int_0^1 X^2(x)\mathrm{d}x = 1. \tag{34}$$

For nonlinear differential Eq. (30) the initial conditions are

$$T(0) = A, \quad \dot{T}(0) = 0. \tag{35}$$

Equation (30) with initial conditions (35) is a second order forced nonlinear differential equation for which it is very difficult to find an exact solution. In what follows, for Eqs. (30) and (35) we will apply OAFM.

3. THE OPTIMAL AUXILIARY FUNCTIONS METHOD (OAFM)

In what follows we consider a general nonlinear differential equation

$$L[T(t)] + N[T(t)] = 0, \quad t \in D, \tag{36}$$

with the initial conditions

$$B\left[T(t), \frac{\mathrm{d}T(t)}{\mathrm{d}t}\right] = 0, \tag{37}$$

in which L and N are linear and nonlinear operators, respectively, D is the domain of interest, B is a boundary operator. The approximate solution of Eq. (36) is supposed to be

$$\bar{T}(t) = T_0(t) + T_1(t, C_i), \quad i = 1,2,..., n, \tag{38}$$

where C_i are n arbitrary parameters at this moment. The initial approximation T_0 and the first approximation T_1 can be obtained from the linear differential equations (for details see [20–25]:

$$L[T_0(t)] = 0, \quad B\left[T_0(t), \frac{\mathrm{d}T_0(t)}{\mathrm{d}t}\right] = 0, \tag{39}$$

$$L[T_1(t)] + N[T_0(t)] + \sum_{k \geq 1} \frac{T_1^k(t, C_i)}{k!} N^{(k)}[T_0(t)] = 0, \tag{40}$$

but with the following crucial remarks.

From the linear differential Eq. (39) the solution $T_0(t)$ is known and therefore can be expressed as

$$\bar{T}_0(t) = \sum_{l=1}^{p} a_l f_l(t), \tag{41}$$

in which the coefficients a_e, the functions $f_e(t)$ and the integer p are well-known. On the other hand, the nonlinear operator $N[T_0(t)]$ calculated for $T_0(t)$ given from the above equation may be written as

$$N[T_0(t)] = \sum_{j=1}^{s} b_j g_j(t)\,, \tag{42}$$

where the coefficients b_j, the functions $g_j(t)$ and positive integer s are clear known, and depend on the initial approximation $T_0(t)$ and also on the nonlinear operator N.

Unfortunately Eq. (40) is very difficult to be solved, such that this nonlinear differential equation is replaced with the following linear differential equation

$$\begin{aligned} &L[T_1(t, C_i)] = \sum_{k=1}^{r} A_k(t, g_j, f_l, C_i), \\ &B\left(T_1(t, C_i), \frac{\mathrm{d}T_1(t, C_i)}{\mathrm{d}t}\right) = 0, \end{aligned} \tag{43}$$

where A_k are r so-called auxiliary functions which depend on n arbitrary parameters and on the functions f_l and g_j known from Eqs.(41) and (42). The expression (43) is not unique. Finally, the unknown parameters C_i, i=1,2,…,n can be optimally identified by means of rigorous mathematical procedures such as the least square method, Ritz method, collocation method, Galerkin method and so on. Our proper technique leads to very accurate results, without the presence of any small parameters in the governing equations or in the boundary/initial conditions.

4. APPLICATION OF THE OAFM TO GOVERNING NONLINEAR EQUATION OF NANOBEAM

To find an analytical approximate solution for nonlinear differential Eqs. (30) and (35) near the primary resonance, we appeal to OAFM taking into account that $\omega \approx \nu\pi$. Making the transformation

$$\tau = \Omega t\,, \quad T(t) = A\theta(\tau)\,, \tag{44}$$

Equations. (30) and (35) can be rewritten as

$$\theta'' + \left(\frac{\omega}{\Omega}\right)^2 \theta + \frac{aA^2}{\Omega^2}\theta^3 = \frac{f_0}{A\Omega^2}\sin\frac{\omega\tau}{\Omega} = 0\,, \quad \theta(0) = 1, \theta'(0) = 0\,, \tag{45}$$

where the prime denotes derivatives with respect to τ and Ω is the frequency of the system.

The linear and nonlinear operators corresponding to Eq. (45) are respectively

$$L[T(\tau)] = \theta'' + \theta; \quad N[\theta(\tau)] = \left(\frac{\omega^2}{\Omega^2} - 1\right)\theta(\tau) + \frac{aA^2}{\Omega^2}\theta^3 - \frac{f_0}{A\Omega^2}\sin\frac{\omega\tau}{\Omega}. \tag{46}$$

The approximate solution of Eq. (45) can be written as

$$\bar{\theta}(\tau) = \theta_0(\tau) + \theta_1(\tau). \tag{47}$$

The initial approximate solution $\theta_0(\tau)$ is determined from the linear differential equation

$$\theta_0''(\tau) + \theta_0(\tau) = 0, \quad \theta_0(0) = 1, \quad \theta_0'(\tau) = 0, \tag{48}$$

whose solution is

$$\theta_0(\tau) = \cos\tau. \tag{49}$$

Inserting Eq. (49) into the second expression of Eq. (46) it holds that

$$N[\theta_0(\tau)] = N_1\cos\tau + N_3\cos 3\tau + N_4\sin\frac{\omega}{\Omega}\tau, \tag{50}$$

where

$$N_1 = \frac{\omega^2}{\Omega^2} - 1 + \frac{3aA^2}{4\Omega^2}; \quad N_3 = \frac{aA^2}{4\Omega^2}; \quad N_4 = -\frac{f_0}{A\Omega^2}. \tag{51}$$

From Eqs. (49), (50) and (43) we propose the following equation for the first approximation

$$\begin{gathered}\theta_1''(\tau) + \theta_1(\tau) = (C_1 + 3C_2\cos 2\tau + 2C_3\cos 4\tau)(N_1\cos\tau + \\ +N_3\cos 3\tau), \ \ \theta_1(0) = \theta_1'(\tau) = 0.\end{gathered} \tag{52}$$

After some manipulations the Eq. (52) may be rewritten as

$$\begin{gathered}\theta_1'' + \theta_1 = [(C_1 + C_2)N_1 + (C_2 + C_3)N_3]\cos\tau + \\ +[(C_2 + C_3)N_1 + C_1N_3]\cos 3\tau + (C_2N_3 + C_3N_1)\cos 5\tau + \\ +C_3N_3\cos 7\tau.\end{gathered} \tag{53}$$

Avoiding the secular term into Eq. (53) we can find the frequency of the system

$$\Omega^2 = \omega^2 + \frac{3a}{4}A^2 + \frac{aA^2}{4}\frac{C_2 + C_3}{C_1 + C_2}. \tag{54}$$

The solution of Eq. (52) becomes

$$\theta_1(\tau)=\frac{(C_2+C_3)N_1+C_1N_3}{8}(\cos\tau-\cos 3\tau)+\frac{C_2N_3+C_3N_1}{24}\cdot \\ \cdot(\cos\tau-\cos 5\tau)+\frac{C_3N_3}{48}(\cos\tau-\cos 7\tau). \tag{55}$$

The approximate solution of Eqs. (30) and (35) can be obtained from Eqs. (44), (45), (49) and (55):

$$T(t)=A\cos\Omega t+\frac{A[(C_2+C_3)N_1+C_1N_3]}{8}(\cos\tau-\cos 3\tau)+ \\ +\frac{A(C_2N_3+C_3N_1)}{24}(\cos\tau-\cos 5\tau)+ \\ +\frac{AC_3N_3}{48}(\cos\tau-\cos 7\tau). \tag{56}$$

where Ω is given by Eq. (54) and the convergence-control parameters are determined using a collocation approach.

5. NUMERICAL APPLICATION

The efficiency of our procedure can be proved through the following particular case which is obtained for K_p= 0.05, K_L= 0.2, K_{NL}= 0.09, γ = 0.3, α = 0.25.

The coefficients of Eq. (30) are obtained from Eq. (31):

$$\varpi = 1.1;\ f = 0.003;\ a = 0.216;\ v = 0.35.$$

The values of the convergence-control parameters C_i obtained by the above-mentioned method are C_1=4.363742866, C_2=–7.244001113, C_3=7.130479859, Ω = 1.172236064.

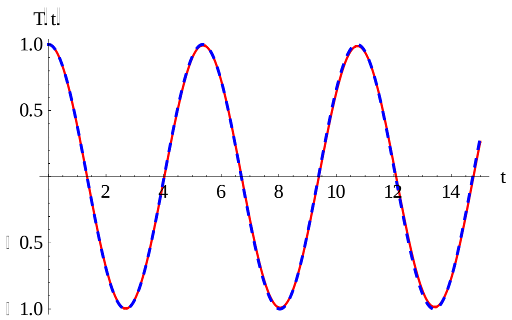

Fig. 2 – Comparison between the analytical solution (56) and numerical integration results for Eqs. (30) and (35) : ______ numerical solution; _ _ _ _ approximate solution.

Figure 2 shows the comparison between approximate solution (56) and numerical solution obtained by means of a fourth order Runge-Kutta approach.

It can be observed that our approximate solution for the nanobeam obtained through OAFM is nearly identical to the numerical integration results, which proves the efficiency of our analytical technique.

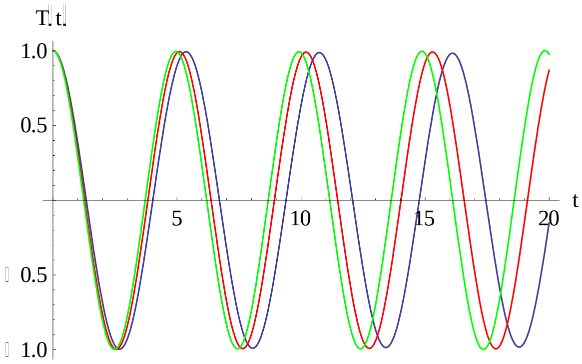

Fig. 3 – The effect of the parameter K_{NL} on the vertical displacement:
blue = 0.2; red = 0.3; green = 0.4.

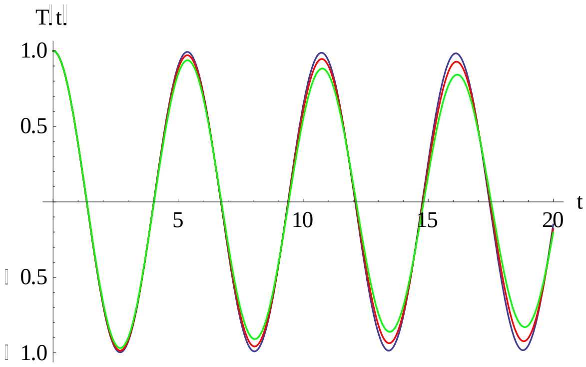

Fig. 4 – The effect of the parameter f on the vertical displacement:
blue = 0.003; red = 0.013; green = 0.028.

Figures 3 and 4 show the effect of the parameters K_{NL} and f, respectively on the vertical displacement of the nanobeam.

From Figure 3 we deduce that if the nonlinear coefficient of Winkler-Pasternak foundation increases, then the period decreases, but the amplitude of the vibration remains unchanged.

From Figure 4 it is clear that if the parameter f increases, then the period remains unchanged but the amplitude of the vibration decreases.

6. CONCLUSIONS

According to the present results, the oscillatory behavior of simply supported flexible uniform nanobeam, taking into consideration the curvature of the beam is studied. The Euler-Bernoulli beam is subjected to a mechanical impact modeled by Dirac-delta function and to the nonlinear Winkler-Pasternak foundation. The OAFM procedure was applied to solve the complex nonlinear differential equation introducing so-called auxiliary functions and some convergence-control parameters, without supplementary hypotheses. These parameters are optimally determined by a rigorous mathematical procedure. Our technique leads to a very accurate solution using only the first iteration. It should be emphasized that any nonlinear dynamical system is reduced to only two linear differential equations. The effect of different parameters on the amplitude and period of vibration of the nanobeam are analyzed.

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