ON THE UNIAXIAL DEFORMATION OF NONHOMOGENEOUS MATERIALS

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Abstract. The aim of the paper is to find a parametrical representation for the constitutive laws for which the motion equations can be associated to a pseudospherical surface with negative Gaussian curvature K. These constitutive laws suggest the existence of a Tzitzeica surface for which the ratio K/d^4 is constant, where d is the distance from the origin to the tangent plane at an arbitrary point.

Key words: Tzitzeica surface, constitutive law, nonhomogeneous materials.

1. INTRODUCTION

The formulation of the constitutive property for a material has to describe its global mechanical properties which cannot be treated empirically [1, 2]. The paper tries to determine a class of constitutive laws from the pseudospherical reduction method for a given [3–5]. This procedure associates the motion equations to a pseudospherical surface Σ with negative Gaussian curvature K. For K/d^4 = const., a Tzitzeica surface is obtained [6–8]. Here, d is the distance from the origin to the tangent plane at an arbitrary point. Tzitzeica surfaces are analogues to the spheres in affine differential geometry invariants under the centro-affine transformations. The parametrical representation for the constitutive laws for which the motion equations attached to the material system can be associated to a pseudospherical surface is a new approach which may improve the estimation of structurally properties of a material.

2. THE PSEUDOSPHERICAL REDUCTION OF THE PROBLEM

For the 1D problem of uniaxial deformation of a nonhomogeneous beam, the governing equations in a Lagrangian system of coordinates (X,t) are written as [3, 4]

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$$\varepsilon_t = v_X, \quad \rho_0 v_t = \sigma_X. \tag{1}$$

A general form for the constitutive law may be written as

$$\sigma = \sigma(\varepsilon, X), \tag{2}$$

where σ and ρ are the uniaxial stress and the density of the material, respectively. The stretch is $\varepsilon = \rho_0 / \rho - 1$, the density of the material in the underformed state is ρ_0 and the material velocity is written as $v(X,t)$. In the Eulerian coordinates $x = x(X,t)$, we get

$$\mathrm{d}x = (\varepsilon + 1)\mathrm{d}X = v\mathrm{d}t, \quad \rho_0 \mathrm{d}X = \rho \mathrm{d}x - \rho v \mathrm{d}t, \tag{3}$$

where X corresponds to the particle function ψ of the Martin formulation [3, 4]. The independent variables are σ and ψ. For $\rho_0 = 1$, the Monge–Ampère equation is obtained

$$\xi_{\sigma\sigma}\xi_{\psi\psi} - \xi_{\sigma\psi}^2 = \varepsilon_\sigma, \tag{4}$$

with

$$t = \xi_\sigma, \quad v = \xi_\psi, \quad \mathrm{d}x = \xi_\psi \xi_{\sigma\sigma} + (\xi_\psi \xi_{\sigma\psi} + \varepsilon)\mathrm{d}\psi,$$
$$0 < |\, \xi_{\sigma\sigma}\varepsilon \,| < \infty.$$

The particle trajectories are calculated from

$$x = \int [\xi_\psi \xi_{\sigma\sigma} + (\xi_\psi \xi_{\sigma\psi} + \varepsilon)\mathrm{d}\psi], \quad t = \xi_p, \tag{5}$$

in terms of σ, for $\psi = \text{const.}$, if a solution of (4) is known [8]. By solving (5), we obtain the solution $\sigma(\psi, t)$. The solution of (1)–(2) is parametrically determined with respect to the Lagrangian variables $x = x(\psi, t)$, $v = v(\psi, t)$, $\sigma = \sigma(\psi, t)$. To connect this problem to a surface Σ in R^3, we write the surface in the Monge parameterisation $r = xe_1 + ye_2 + z(x,y)e_3$, with $r = r(x,y,z)$ the position vector of a point P on the surface.

The first and second fundamental forms are defined as

$$\begin{gathered} I = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2 = (1 + z_x^2)\mathrm{d}x^2 + 2z_x z_y \mathrm{d}x\mathrm{d}y + (1 + z_y^2)\mathrm{d}y^2, \\ II = e\mathrm{d}x^2 + 2f\mathrm{d}x\mathrm{d}y + g\mathrm{d}y^2 = \left(1 + z_x^2 + z_y^2\right)^{-1/2}(z_{xx}\mathrm{d}x^2 + 2z_{xy}\mathrm{d}x\mathrm{d}y + z_{yy}\mathrm{d}y^2). \end{gathered} \tag{6}$$

The Gaussian curvature of Σ is given by

$$\mathrm{K} = (eg - f^2)(EG - F^2)^{-1} = -(z_{xx}z_{yy} - z_{xy}^2)(1 + z_x^2 + z_y^2)^{-2}.$$

If Σ is a hyperbolic surface, total curvature is negative and the asymptotic lines on Σ are the parametric curves. For the same independent variables as before, we have σ and ψ with $\sigma = z_x$, $\psi = z_y$. The dependent variable ξ yields to $\xi_\sigma = x$, $\xi_\psi = y$. Therefore [5–8]

$$\xi_{\sigma\sigma} = z_{yy}(z_{xx}z_{yy} - z_{xy}^2)^{-1}, \quad \xi_{\psi\psi} = z_{xx}(z_{xx}z_{yy} - z_{xy}^2)^{-1},$$
$$\xi_{\sigma\psi} = z_{xy}(z_{xx}z_{yy} - z_{xy}^2)^{-1}.$$

The Gaussian curvature (6) yields

$$K = (1 + \sigma^2 + \psi^2)^{-2}(\xi_{\sigma\sigma}\xi_{\psi\psi} - \xi_{\sigma\psi}^2)^{-1}.$$

The Gaussian curvature may be set into correspondence with the Martin's Monge–Ampère equation (4) by writing

$$\varepsilon_\sigma = K^{-1}(1 + \sigma^2 + \psi^2)^{-2}, \tag{7}$$

for

$$K = A^2(1 + \sigma^2 + X^2)^{-2} \tag{8}$$

with A the Lagrangian wave for which $A^2 = \dfrac{\partial\sigma}{\partial\varepsilon|_X}$. The surface Σ is restricted to be a Ţiţeica surface if $K = -a^{-2}$, $a = \text{const}$. The relation (8) gives

$$\frac{\partial^2\sigma}{\partial\varepsilon^2|_X} = 2a^{-2}(1 + \sigma^2 + X^2)\sigma\frac{\partial\sigma}{\partial\varepsilon}|_X > 0, \quad \sigma > 0. \tag{9}$$

By integrating (9) we get

$$\varepsilon = \frac{1}{2}a^2(1 + X^2)^{-3/2}\Big(\arctan\big(\sigma(1 + X^2)^{-1}\big) + \\ + \sigma(1 + \sigma^2 + X^2)^{-1}\sqrt{1 + X^2}\Big) + \alpha(X), \tag{10}$$

with $\alpha(X)$ arbitrary. For $\sigma|_{\varepsilon=0} = 0$, it results $\alpha(X) = 0$.

The (10) represents a class of constitutive laws for which (1) is associated to a pseudospherical surface Σ. Several constitutive laws can be derived for specified practical problems, starting from (10).

If introduce into (9) the stress representation

$$\sigma = \sqrt{1+X^2}\tan\left(a^{-1}\sqrt{1+X^2}(c-c_0)\right), \tag{11}$$

we obtain

$$\begin{aligned}\varepsilon = \frac{1}{2}a^2(1+X^2)^{-1}\left(a^{-1}(c-c_0)+(1+X^2)^{-1/2}\right)\cdot \\ \cdot\sin\left(2a^{-1}\sqrt{1+X^2}(c-c_0)\right).\end{aligned} \tag{12}$$

Relations (11) and (12) represent a parametric representation for the constitutive laws $\sigma = \sigma(\varepsilon, X)$ for which the equations (1) are associated to a pseudospherical surface Σ.

These equations lead to $\sigma_{XX} = \varepsilon_{tt}$ by using (7)

$$\sigma_{XX} = \left(a^2(1+\sigma^2+X^2)^{-2}\sigma_t\right)_t . \tag{13}$$

3. CONSTITUTIVE LOW

Damaged materials form a class of unusual elastic materials that show extreme nonlinearity, hysteresis and discrete memory. This class includes pearlitic steel, fibre-reinforced metal matrix composites, cement, concrete, ceramics, rocks, sand, soil etc. A damaged material is a consolidated one, an aggregate of grains fused together by temperature, pressure and chemical processes. The rock specimens with different ratios of length to diameter, their initial stiffness values of are unchanged, but the post-peak stiffness increases as the ratios of length to diameter increase .

As pointed out, we consider some materials as the Berea sandstone, the Kayenta sandstone [9], the marble specimen [10], the discontinuous random Polyethylene fibre reinforced cement [11] and some rocks [12–14]. For these materials, the slow dynamics is manifested by a significant alteration of the dissipation into the material. This alteration pulls after itself the disturbance of the modulus after mechanical disturbance, a memory of the disturbed strain state. The modulus and wave dissipation progressively recover to their original values after $10^3 - 10^4$ seconds. Slow dynamics is a sensitive probe of the micromechanics of the system [15–17]. These materials, Fig. 1, are aggregate of grains which act as rigid vibrating units, while the contacts between them – the bond system – constitute a set of interfaces that control the behaviour of the material. The interfaces have a typical size of $1\,\mu\text{m}$ [18, 19].

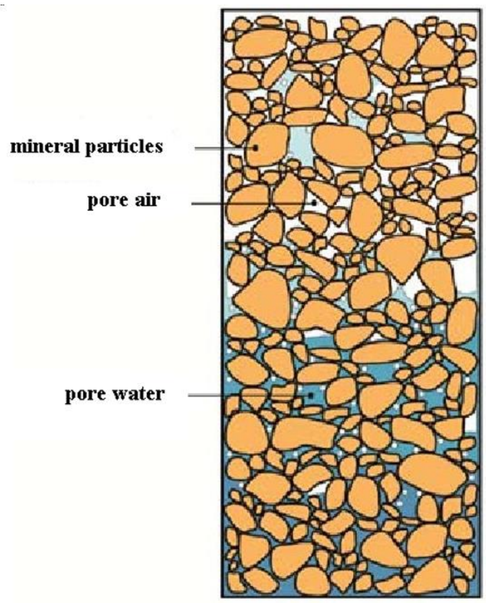

Fig. 1 – Schematic show of a granular material.

To identify the behavior of these materials, according to (11), (12), we consider a general constitutive law under the form

$$\sigma = \sum_{m=1}^{M} C_m \sqrt{1+X^2}\tan\left(a_m^{-1}\sqrt{1+X^2}(c_m - c_{0m})\right), \tag{14}$$

and

$$\begin{aligned}\varepsilon = \frac{1}{2}\sum_{m=1}^{M} B_m a_m^2 (1+X^2)^{-1}\left(a_m^{-1}(c_m - c_{0m}) + (1+X^2)^{-1/2}\right)\cdot \\ \cdot \sin\left(2a_m^{-1}\sqrt{1+X^2}(c_m - c_{0m})\right).\end{aligned} \tag{15}$$

The unknown parameters $p_m = \{C_m, B_m, a_m, c_m, c_{0m}\}$, $m = 1,...,M$, are written for each m as

$$M_{ij} = (i-1)JKLQ + (j-1)KLQ + (k-1)LQ + (l-1)Q + q,$$

with I, J, K, L and Q are total number of values associated to each parameter p_m. This number is counted from the first set of $p = \{p_m\}$.

The inverse problem is solved by a genetic algorithm. A square sum of differences between the measured stress-strain results for some selected cases of uniaxial tension problem for non-homogeneous materials is considered as

$$W = \sum_{n=1}^{N} (\sigma_i^n - \bar{\sigma}_i^n)^2, \tag{16}$$

where $\bar{\sigma}_i^n$ are the measured stress at point n on the strain-stress diagram $\sigma - \varepsilon$, σ_i^n denotes the computed strain-stress $\sigma - \varepsilon$ at the same point n, and N is the number of points from the measured uniaxial diagram $\sigma - \varepsilon$. The fitness $\tilde{F}$ is defined as a reciprocal number of the function W

$$\tilde{F} = \sum_{n=1}^{N} (\bar{\sigma}_i^n)^2 / W. \tag{17}$$

The convergence criterion is related to the expression Z

$$Z = \frac{1}{2}\log_{10}\frac{W}{W_0}. \tag{18}$$

In the following, we assume $M = 2$, $N = 90$, the number of populations is 25, ratio of reproduction 1, number of multi-point crossovers 1, probability of mutation 0.25 and the maximum number of generations 300.

4. RESULTS

The constitutive laws for three Berea sandstone samples are illustrated in Fig. 2. For the curve 3, the stress marks a peak representing the failure. The curve 2 shifts the macroscopic failure mode from brittle to ductile. At the curve 3, the strain hardening persists up. This behavior [18,19] is qualitatively the same to the experimental results reported in [9,10]. The failure means to be a function of the parameters a_m, for specified other parameters. The inelastic behavior is macroscopically ductile, and the conventional approach is to pick its failure strength to be the stress level at an arbitrary axial strain. The strength so determined is compiled as a function of a_1 and a_2, for specified c_m, c_{0m}, $m = 1,2$. In [9], the effective pressure plays the same role as the parameters a_m, $m = 1,2$.

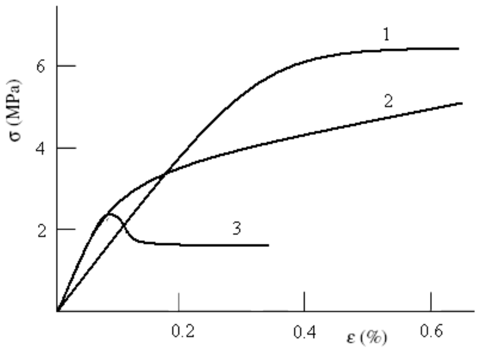

Fig. 2 – The stress-strain law for Berea sandstone.

The constitutive law for four Kayenta sandstone samples are illustrated in Fig. 3. For the curve 3, the stress marks a failure peak. For curves 3 and 2, the macroscopic failure mode is changed from brittle to ductile. At the curve 1, the strain hardening is observed. This behavior is qualitatively the same to the experimental results reported in [9]. Also, the failure means to be a function of the parameter a_m, $m=1,2$.

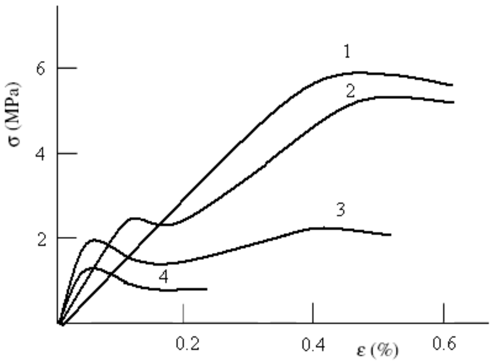

Fig. 3 – The stress-strain law for Kayenta sandstone.

It is well known that the mechanical properties of rock or marble specimens are scale dependent. It is seen from Fig. 4. that, for the four rock specimens with different ratios of length to diameter (the ratios of length to diameter for specimens 1–4 are 0.5, 1, 2 and 3, respectively), their initial constitutive curve is unchanged, but the post-peak stiffness increases as the ratios of length to diameter increase. This demonstrates that a specimen with a smaller diameter will correspond to a larger post-peak stiffness value [10].

The constitutive law for discontinuous random polyethylene fiber reinforced cement is illustrated in Fig. 5, for the fiber volume fraction 0.1 % and 0.5%, respectively. We see from the curve 2 that the composite with 0.1 % fiber volume fraction have a catastrophic failure. We see a significant ductility and accentuated fracture toughness for 1% fiber volume fraction from the curve 1.

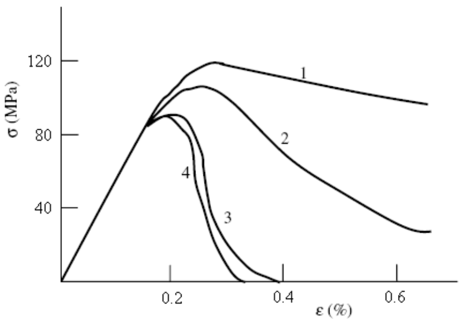

Fig. 4 – Constitutive law for marble specimens with different ratios of length to diameter under axial compression condition.

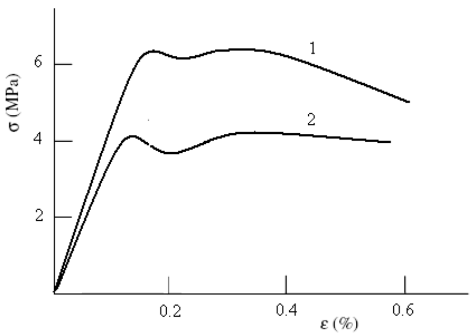

Fig. 5 – Constitutive law for polyethylene fiber reinforced cement for the fiber volume fraction 0.1 % (curve 2) and 0.5% (curve 1).

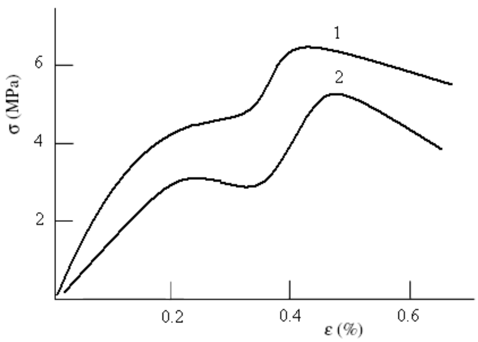

Fig. 6 – Constitutive law for polyethylene fiber reinforced cement for the fiber volume fraction 1 % (curve 2) and 1.5% (curve 1).

Another constitutive law associated to a Tzitzeica surface with K/d^4 constant, is considered for the same data for discontinuous random polyethylene reinforced cement with fibres. The results are presented in Fig. 6, for two values of the fibre volume fraction, i.e. 1% for the curve 2 and 1.5% for the curve 1. A significant ductility and fracture toughness are put into evidence.

5. CONCLUSIONS

The goal of the paper is to find a parametrical representation for a class of constitutive laws for which the motion equations is associated to a pseudospherical surface. The uniaxial deformation problem for non-homogeneous materials is discussed via the pseudospherical reduction technique. A genetic algorithm is performed to study some inverse problems associated to experimental results. For Berea sandstone and Kayenta sandstone, the strength of material can be determined as a function of a_1 and a_2, the relation to the Gaussian curvature being $K=-\frac{1}{a_1^2+a_2^2}$. So, we can conclude that if the motion equations can be associated to a pseudospherical surface Σ, of Gaussian curvature K, the strength of material can be described as a function of K.

For discontinuous random Polyethylene fiber reinforced cement, the results yield to the conclusion that the parameters c_1 and c_2 are important in describing the stress-strain curve for fiber reinforced composites. We can conclude that this parameter may be proportional to the fiber volume fraction.

A subclass of the constitutive laws is associated to a Tzitzeica surface, for which the ratio K/d^4 (d is the distance from the origin to the tangent plane at an arbitrary point), is constant. For Berea sandstone, the parameters a_1 and a_2 are related to d, and it is important in describing the ductility and fracture toughness properties. The method is restricted to the uniaxial deformations and may be extended to predict the hysteretic behaviour of materials. The Bouc-Wen model is mostly used within the following inverse problem approach: given a set of experimental input–output data [20, 21].

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