DOUBLE PENDULUM'S SLOW DYNAMICS ANALYSIS

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Abstract. By extending the analysis of the slow dynamics, the solutions of a double pendulum motion written as a linear superposition of Coulomb vibrations have broken symmetry concerning the driving excitation. The solutions allow a qualitative analysis not only of the quasi-periodic behavior but also of the chaotic behavior of the pendulum. The solutions permit understanding what causes an oscillation to lose its periodicity and hence become chaotic.

Key words: pendulum motion, Coulomb vibrations, slow dynamics.

1. INTRODUCTION

Slow dynamics is a term concerning the mechanically or thermally induced motion of a mechanical system that displays broken symmetry concerning the driving excitation [1]. For instance, in rock and concrete, the elastic modulus always displays a quasi-static decrease in response to the fast and symmetric oscillatory driving stresses [2–4],

The broken symmetry is observed in sandstones in which the temperature is slightly increased or decreased [5] The stress field in the material caused by the anisotropy of the thermal expansion is extremely inhomogeneous on the grain scale. *i.e*. on length scales of order 100 microns. The observed asymmetries are contrary to other physical systems that display proportion to the $\log(t)$ such as creep in rock and metals, magnetic fields, spin glasses [6], etc., that respond with a symmetric response to the driving stress [7].

Materials with strain memory have in common a small volume of elasticity soft constituents (bonds in synthetic ceramics) where the slow dynamics originate, distributed within a rigid matrix *e.g*. grains in the ceramics.

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In cracked materials, the slow dynamics are localized phenomena. Materials that exhibit slow dynamics are defined over a seemingly large number of scales of the discontinuity features (the elastically soft regions) where the behavior originates.

Clear evidence of slow dynamics in two-phase metal, sintered ceramics, and crystalline rock or granites is demonstrated in the specialized literature [8, 9]. Of significant interest is that we can induce slow dynamics in materials that otherwise exhibit no slow dynamics whatsoever, by introducing a crack. We show clear evidence of slow dynamics in samples of a two-phase metal, a sintered ceramic, and a crystalline rock (granite).

Experiments on the transitory change in wave amplitude correspond to either the application of a mechanical impulse or a large amplitude continuous wave source that progressively recovers with time. We refer to the eigenmode slope amplifier that can be used to amplify the slow dynamical response significantly.

The behavior of the materials where the slow dynamics have never before been observed (*e.g.*, a sintered ceramic; a pearlite/graphite mixed phase metal; coarse and fine-grained granite; and a cracked, mostly pearlite steel) is described in and compared with two of the sedimentary rocks where they have been observed before [8–10]. Figure 1 presents the double pendulum subjected to periodic non-conservative loads.

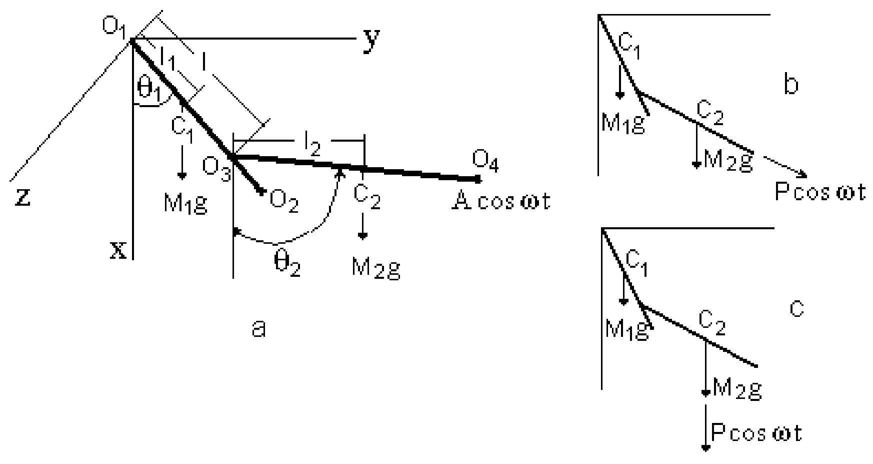

Fig. 1 – Double pendulum subjected to periodic non-conservative loads.

The choice of the pendulum as a model system to be analyzed by the LEM method [11–14] has a strong historical precedent. The pendulum is a standard mechanical example in classical mechanics courses. 400 years after Galileo's initial work, the pendulum has again become an object of research as a chaotic system.

The LEM methods can be extended to the analysis of dynamical systems capable of producing not-so-regular vibrations, because not only the zero-divisor terms but certain small-divisor terms are included in the analysis [15–18]. The Lie transformation method applies to the Bolotin systems of equations, but is not adequate for the more complex systems as the equations of a double pendulum, because of the non-standard type of involved nonlinearities).

The pendulum would seem to be one of the simplest physical systems. But its behavior can be rich and complex. With the recent interest in chaotic motions of non-linear systems, it seems appropriate to look into the possible existence of chaos in this particular system. We will see in this paper how the study of the free and forced motion of the pendulum can be facilitated by using the linear equivalence method The LEM representations of the non-simple-regular solutions of the pendulum are describable as a linear superposition of Coulomb vibrations [14]. The solutions are designed to establish some qualitative conclusions, relevant for describing the correct dynamical behavior of the pendulum.

The method LEM [8–12] is applicable for the integration of first-order differential equations having algebraic nonlinearities and an arbitrary number of unknowns. For the case of Hamiltonian systems LEM is consistent with the KAM (Kolmogorov-Arnold-Moser) theory, which gives the limits of integrability for such systems. In contrast to other methods, the LEM method is not limited to conservative systems.

The general theory of integrable conservative systems due to Hamilton-Jacobi [9] predicts only simple harmonic limit-cycle solutions. Experimental observation of fracton mode in one-dimensional Cantor composites is done in [15] and the threshold lowering for subharmonic generation in Cantor composite structures in [16–18]. Oscillation modes in a surface shear wave propagating through a regularly laminated electro-elastic half-space are presented in [19] and the subharmonic generation in piezoelectrics with Cantor-like structure in [20]. The soliton and cnoidal theories are presented in [21] and different inverse problems solved by using the cnoidal theory in [22].

2. EQUATIONS OF MOTION

Figure 1 shows a double pendulum subjected to the periodic non-conservative loads [12]. It is consisted from two straight rods O_1O_2 and O_3O_4 of masses M_1 , M_2, lengths $2l_1$, $2l_2$, and mass centres C_1, C_2. The rods are articulated in O_3 and suspended in O_1, so that they can move in the vertical plane

xO_1y without friction. Other notations from Fig. 1 are: $l = O_1O_3$, $l_1 = O_1C_1$, $l_2 = O_3C_2$. We note by θ_1, θ_2 the displacement angles in rapport to the vertical O_1x, I_1 the mass moment of inertia of O_1O_2 with respect to C_1, I_2 the mass moment of inertia of O_3O_4 with respect to C_2, and g the gravitational constant.

The forces acting upon the pendulum are, firstly, the weights of bars. The generalised forces are

$$\begin{gathered} G_1 = -M_1 l_1 g \sin\theta_1 - M_2 g l \sin\theta_1, \\ G_2 = -M_2 g l_2 \sin\theta_2. \end{gathered} \tag{1}$$

Secondary, we consider three cases of non-conservative forces:

Case A – a generalised force $A\cos\omega t$ in O_4, $[A] = \dfrac{\mathrm{kg}\times\mathrm{m}^2}{\mathrm{s}^2}$, Fig. 1a.

Case B – a force $P\cos\omega t$ along O_3O_4, $[P] = \dfrac{\mathrm{kg}\times\mathrm{m}^2}{\mathrm{s}^2}$, Fig. 1b.

Case C – a vertical force $P\cos\omega t$ in C_2, Fig. 1c.

Therefore, the generalised forces are

$$\begin{gathered} Q_1 = \begin{cases} 0 & \text{for case } A \\ Pl\sin(\theta_2 - \theta_1)\cos\omega t & \text{for case } B \\ -Pl\sin\theta_1\cos\omega t & \text{for case } C \end{cases} \\ Q_2 = \begin{cases} A\cos\omega t & \text{for case } A \\ 0 & \text{for case } B \\ -Pl_2\sin\theta_2\cos\omega t & \text{for case } C \end{cases} \end{gathered} \tag{2}$$

The motion equations obtained from the Lagrange equations

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot\theta_1}\right) - \frac{\partial T}{\partial\theta_1} = G_1 + Q_1, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot\theta_2}\right) - \frac{\partial T}{\partial\theta_2} = G_2 + Q_2, \end{gathered} \tag{3}$$

are written in the form

$$\begin{cases} \ddot\theta_1 + \alpha[\ddot\theta_2\cos(\theta_2 - \theta_1) - \dot\theta_2^2\sin(\theta_2 - \theta_1)] + \beta\sin\theta_1 = q_1, \\ \ddot\theta_2 + \gamma[\ddot\theta_1\cos(\theta_2 - \theta_1) + \dot\theta_1^2\sin(\theta_2 - \theta_1)] + \sin\theta_2 = q_2, \end{cases} \tag{4}$$

where dimensionless variables and coefficients are given by

$$
\begin{aligned}
& t \to t\sqrt{\eta}, \quad \omega \to \frac{\omega}{\sqrt{\eta}}, \quad \eta = \frac{gM_2 l_2}{I_2 + l_2^2 M_2}, \\
& \alpha = \frac{M_2 l l_2}{I_1 + l_1^2 M_1 + l^2 M_2}, \ \gamma = \frac{M_2 l l_2}{I_2 + l_2^2 M_2}, \\
& \beta = \frac{M_1 l_1 + M_2 l}{I_1 + l_1^2 M_1 + l^2 M_2} \cdot \frac{I_2 + l_2^2 M_2}{M_2 l_2}, \\
& \delta = \frac{A}{gM_2 l_2}, \ \tilde{\mu} = \frac{P}{gM_2}, \ \nu = \frac{4}{3}\alpha\tilde{\mu}.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
q_1 &= \begin{cases} 0 & \text{for case } A \\ \nu \sin(\theta_2 - \theta_1)\cos\omega t & \text{for case } B \\ -\nu \sin\theta_1 \cos\omega t & \text{for case } C \end{cases} \\
q_2 &= \begin{cases} \delta \cos\omega t & \text{for case } A \\ 0 & \text{for case } B \\ -\tilde{\mu} \sin\theta_2 \cos\omega t & \text{for case } C \end{cases}
\end{aligned} \tag{6}
$$

The dot represents the differentiation with respect to the nondimensional variable t.

Setting

$$
m = \frac{M_1}{M_2}, \ r = \frac{l}{l_1}, \quad s = \frac{l_2}{l_1}, \tag{7}
$$

the motion of the double pendulum depends on the control parameters r, s, m, δ, $\tilde{\mu}$ and ω. It is simple to show that α, β, γ become

$$
\alpha = \frac{rs}{r^2 + \frac{4}{3}m}, \ \beta = \frac{4}{3}s \cdot \frac{r+m}{r^2 + \frac{4}{3}m}, \quad \gamma = \frac{3}{4}\frac{r}{s}. \tag{8}
$$

By introducing the new variables

$$
\begin{gathered}
\theta_1 = z_1, \quad \theta_2 = z_2, \ \dot{\theta}_1 = z_3, \quad \dot{\theta}_2 = z_4, \\
\cos\omega t = z_5, \ -\omega \sin\omega t = z_6,
\end{gathered} \tag{9}
$$

the equations (4) are rewritten in the state-space form as

$$\begin{aligned}
&\dot{z}_1 = z_3,\\
&\dot{z}_2 = z_4,\\
&\dot{z}_3 = \frac{1}{1-\alpha\gamma\cos^2(z_2-z_1)}[-\beta\sin z_1 + \alpha\sin z_2\cos(z_2-z_1) +\\
&\quad +\alpha z_4^2\sin(z_2-z_1) + \alpha\gamma z_3^2\sin(z_2-z_1)\cdot\cos(z_2-z_1)\\
&\quad - A_3(z) + \alpha A_4(z)\cos(z_2-z_1)],\\
&\dot{z}_4 = \frac{1}{1-\alpha\gamma\cos^2(z_2-z_1)}[-\sin z_2 + \beta\gamma\sin z_1\cos(z_2-z_1) -\\
&\quad -\gamma z_3^2\sin(z_2-z_1) - \alpha\gamma z_4^2\sin(z_2-z_1)\cos(z_2-z_1) - A_4(z) +\\
&\quad +\gamma A_3(z)\cos(z_2-z_1)],\\
&\dot{z}_5 = z_6,\\
&\dot{z}_6 = -\omega^2 z_5,
\end{aligned} \tag{10}$$

with

$$A_3(z) = \begin{cases} 0 & \text{for case } A\\ -\frac{4}{3}\alpha\tilde{\mu}z_5\sin(z_2-z_1) & \text{for case } B\\ \frac{4}{3}\alpha\tilde{\mu}z_5\sin z_1 & \text{for case } C \end{cases} \tag{11}$$

$$A_4(z) = \begin{cases} -\delta z_5 & \text{for case } A\\ 0 & \text{for case } B\\ \tilde{\mu}z_5\sin z_2 & \text{for case } C \end{cases}$$

We rewrite the system of equations (10) in the following form

$$\dot{z}_n = A_{np}z_p + \frac{g_n(z)}{h_n(z)},\quad n = 1,2,...,6, \tag{12}$$

with

$$h_n(z) = 1 - H_{npq}\cos^2(z_p - z_q) \tag{13}$$

and $h_n \neq 0$. It is assumed to be valid the summation law with respect to repeated indices $(n,p,q,r = 1,2,3,...,6)$. The constants are

$$\begin{aligned}
&A_{13}=1,\quad A_{24}=1\ ,\ A_{56}=1\ ,\ A_{65}=-\omega^2,\\
&B_{45}=1,\ \ C_{31}=-\beta,\quad C_{42}=-1,\\
&D_{412}=\beta\gamma\ ,\ D_{321}=\alpha,\\
&E_{3521}=-\alpha,\\
&F_{3421}=\alpha,\quad F_{4321}=-\gamma,\\
&G_{3321}=\alpha\gamma,\quad G_{4421}=-\alpha\gamma,\\
&H_{321}=\alpha\gamma,\quad H_{421}=\alpha\gamma,\\
&L_{3214}=\alpha,\quad L_{4214}=\gamma,
\end{aligned} \tag{14}$$

and the rest nule.

Next step of the analysis is to introduce the simplification

$$1/[(1-\alpha\gamma\cos^2(z_2-z_1)]\approx 1+\alpha\gamma\cos^2(z_2-z_1)\,. \tag{15}$$

Accordingly, the simplified system of equations are

$$\begin{aligned}
&\dot z_1=z_3,\\
&\dot z_2=z_4,\\
&\dot z_3=-\beta\sin z_1+\alpha\sin z_2\cos(z_2-z_1)-\\
&-\alpha z_4^2\sin(z_2-z_1)+\alpha\gamma z_3^2\sin(z_2-z_1)\cos(z_2-z_1)-\\
&-\alpha\gamma\beta\sin z_1\cos^2(z_2-z_1)+\alpha^2\gamma\sin z_2\cos^3(z_2-z_1)+\\
&+\alpha^2\gamma z_4^2\cos^2(z_2-z_1)\sin(z_2-z_1)+\\
&+\alpha^2\gamma^2 z_3^2\sin(z_2-z_1)\cos^3(z_2-z_1)-A_3(z)-\\
&-\alpha\gamma A_3(z)\cos^2(z_2-z_1)+\\
&+\alpha A_4(z)\cos(z_2-z_1)+\alpha^2\gamma A_4(z)\cos^3(z_2-z_1),\\
&\dot z_4=-\sin z_2+\beta\gamma\sin z_1\cos(z_2-z_1)-\gamma z_3^2\sin(z_2-z_1)-\\
&-\alpha\gamma z_4^2\sin(z_2-z_1)\cos(z_2-z_1)-\alpha\gamma\sin z_2\cos^2(z_2-z_1)-\\
&-\alpha^2\gamma^2 z_4^2\sin(z_2-z_1)\cos^3(z_2-z_1)+\\
&+\alpha\gamma^2\beta\sin z_1\cos^3(z_2-z_1)-\alpha\gamma^2 z_3^2\sin(z_2-z_1)\cos^2(z_2-z_1)-\\
&-A_4(z)-\alpha\gamma A_4(z)\cos^2(z_2-z_1)+\\
&+\gamma A_3(z)\cos(z_2-z_1)+\alpha\gamma^2 A_3(z)\cos^3(z_2-z_1)\\
&\dot z_5=z_6\\
&\dot z_6=-\omega^2 z_5
\end{aligned} \tag{16}$$

In this paper we apply LEM to the simplified system of equations (16) that are written in the form

$$\dot{z}_n = A_{np} z_p + B_n(z) g_n(z)\,, \tag{17}$$

with

$$B_n(z) = 1 + H_{npq}\cos^2(z_p - z_q)\,. \tag{18}$$

The linearised form of equations (18) is

$$\dot{z} = Az\,, \tag{19}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\beta\varsigma & \alpha\varsigma & 0 & 0 & -\alpha\varsigma\delta & 0 \\ \beta\gamma\varsigma & -\varsigma & 0 & 0 & \varsigma\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{pmatrix}, \tag{20}$$

where $\varsigma = 1 + \alpha\gamma$. The characteristic equation $\left(A - \lambda I\right) = 0$,

$$[\lambda^4 + \varsigma(\beta+1)\lambda^2 + \beta\varsigma^2(1-\alpha\gamma)][\lambda^2 + \omega^2] = 0\,, \tag{21}$$

admits the roots $\pm ip_1$, $\pm ip_2$, $\pm i\omega$.

3. THE LINEAR EQUIVALENCE METHOD (LEM)

We begin with a brief explain of LEM in the spirit of [8–12].

Consider a nonlinear Cauchy problem

$$\dot{z} = F\left(z\right),\quad z\left(t_0\right) = z_0\,,\;\; t_0 \in I \subset \mathrm{R}, \tag{22}$$

where F is a vector of n analytic functions $F_j\left(z\right) = \sum_{|v|=1}^{\infty} a_{jv} z^v$ with the coefficients $a_{j\nu} \in I \subset R$. Here $\nu = (\nu_1, \nu_2, \ldots \nu_n)$ are multi-indexes of length n, and $z^v = \prod_{j=1}^{n} z_j^{\nu_j}$.

The LEM method consists in introduction of a new variable $v(t,\sigma)$ defined as an exponential transformation of real parameters σ

$$v(t,\sigma)=e^{<\sigma,z>}, \quad <\sigma,z>=\sum_{j=1}^{n}\sigma_j z_j, \tag{23}$$

where $\sigma=[\sigma_1,\sigma_2,...\sigma_n]\in R^n$.

Using (23), the system (22) is transformed into a partial linear differential equation

$$\frac{\partial v}{\partial t}-\sum_{j=1}^{N_s}\sigma_j F_j(D)v=0, \tag{24}$$

where $F_j(t,D)$ is the formal differential operator associated to $F_j(\sigma)$ i.e.

$$F_j(D)=\sum_{|\nu|=1}^{\infty}a_{j\nu}\frac{\partial^{|\nu|}}{\partial\sigma^{\nu}}..., \quad \frac{\partial^{|\nu|}}{\partial\sigma^{\nu}}=\frac{\partial^{\nu_1+\nu_2...+\nu_n}}{\partial\sigma_1^{\nu_1}\partial\sigma_2^{\nu_2}...\partial\sigma_n^{\nu_n}}. \tag{25}$$

The linear equivalence transformation (23), where z_n, $n=1,2,...,6$ is a solution of (22) with the associated initial conditions, satisfies the partial linear differential equation (24) and initial conditions

$$v(t_0,\sigma_1,\sigma_2,...\sigma_n)=\exp(\sigma_1 z_1^0+\sigma_2 z_2^0+...+\sigma_n z_n^0). \tag{26}$$

We enounce the following results (the proofs are found in [8]):

1. The linear equivalence transformation (23) where $z(t)$ is a solution of (22) satisfies the equation (24) and the initial condition (26) $v(t_0,\sigma)=e^{<\sigma,z_0>}$.

2. If $\sup|a_{j\nu}(t)|<C$, the analytical solution $v\in C^1$ with respect to $t\in I$ of the problem (24), (26), is determined using the known theories of partial linear equations. The solution $v(t,\sigma)$ has the form $v(t,\sigma)=e^{<\sigma,z>}$. Because $z(t)$ satisfies the initial problem (22), the initial solution $z(t)$ is easily obtained from $v(t,\sigma)$.

Now, we apply this method to the nonlinear simplified system of equations (18) with the initial conditions (12).

Let introduce the linear equivalence transformation (LEM) that depends on parameters $\sigma_i\in R$, $i=1,2,3,...,6$

$$v(t,\sigma)=\exp(\sigma_1 z_1+\sigma_2 z_2+\sigma_3 z_3+...+\sigma_6 z_6),\quad \sigma_i\in R. \tag{27}$$

By inserting (27) into (18) we obtain a linear partial differential equation

$$\begin{aligned}\frac{\partial v}{\partial t}&=\sum_{n=1}^{6}\Big(\sum_{p=1}^{6}\sigma_n a_{np}\frac{\partial v}{\partial\sigma_p}+\sum_{p,q=1}^{6}\sigma_n b_{npq}\frac{\partial^2 v}{\partial\sigma_p\partial\sigma_q}+\\&+\sum_{p,q,r=1}^{6}\sigma_n c_{npqr}\frac{\partial^3 v}{\partial\sigma_p\partial\sigma_q\partial\sigma_r}+\\&+\sum_{p,q,r,l=1}^{6}\sigma_n d_{npqrl}\frac{\partial^4 v}{\partial\sigma_p\partial\sigma_q\partial\sigma_r\partial\sigma_l}+\\&+\sum_{p,q,r,l,m=1}^{6}\sigma_n e_{npqrlm}\frac{\partial^5 v}{\partial\sigma_p\partial\sigma_q\partial\sigma_r\partial\sigma_l\partial\sigma_m}\Big),\end{aligned} \tag{28}$$

with the associated initial conditions

$$v(0,\sigma)=\exp(\sigma_1 z_1^0+\sigma_2 z_2^0+...+\sigma_6 z_6^0)\,. \tag{29}$$

We consider for $v(t,\sigma)$ the form

$$\begin{aligned}v(t,\sigma)=1&+\sum_{k=1}^{6}\sum_{i=1}A_k^i\frac{\sigma_k^i}{i!}+\sum_{\substack{k,l=1\\k\neq l}}^{6}\sum_{i,j=1}A_k^iA_l^j\frac{\sigma_k^i\sigma_l^j}{i!j!}+\\&+\sum_{\substack{k,l,m=1\\k\neq l\neq m}}^{6}\sum_{i,j,r=1}A_k^iA_l^jA_r^r\frac{\sigma_k^i\sigma_l^j\sigma_m^r}{i!j!r!}+\\&+\sum_{\substack{k,l,m,n=1\\k\neq l\neq m\neq n}}^{6}\sum_{i,j,r,s=1}A_k^iA_l^jA_r^rA_n^s\frac{\sigma_k^i\sigma_l^j\sigma_m^r\sigma_n^s}{i!j!r!s!}\end{aligned} \tag{30}$$

Inserting (30) into (28) and taking use of (29), the functions $A_n(t)$, $n=1,2,3,...,6$ are determined by an elementary identification procedure (equating the terms of the same power in σ and t). So, we will have

$$A_n(t)=\sum_{k,\eta=0}\{(\mu t)^{k+1}\tilde{A}_{nk}(\eta)\Phi_k(\mu t,\eta)+(\mu t)^k\tilde{B}_{nk}(\eta)\psi_k(\mu t,\eta)\} \tag{31}$$

and $k = 0,1,2,3,...k_{\max}$, $\eta = 0,1,2,3,...,6.$

We determine, via numerical experimentation, which $k_{\max}$ (depending of η) is relevant for capturing the contribution of all nonlinearities of governing equations. So, we assess the efficiency of the method for $k_{\max} = 334$. For $k_{\max} > 334$ the computing complicates without bringing significant terms in solutions. In (31) the functions $\Phi_k(\mu t, \eta)$ and $\psi_k(\mu t, \eta)$ are given by

$$\begin{aligned}\Phi_k(\mu t, \eta) &= \sum_{m=k+1} (\mu t)^{m-k-1} A_m^{(k)}(\eta),\\ \psi_k(\mu t, \eta) &= \sum_{m=k+1} (\mu t)^{m-k-1} B_m^{(k)}(\eta)\end{aligned} \tag{32}$$

and $\mu = \mu(\eta)$

$$\mu = \sum_{j=1}^{6} \alpha_j \lambda_j, \tag{33}$$

with $\sum_{l=1}^{6} \alpha_j = \eta + 1$, $\alpha_j \geq 0$, $j = 1,2,3,...,6$.

The quantities λ_j in (33) are $\lambda_1 = p_1$, $\lambda_2 = -p_1$, $\lambda_3 = p_2$, $\lambda_4 = -p_2$, $\lambda_5 = \omega$, $\lambda_6 = -\omega$, where $\pm ip_1, \pm ip_2$, $\pm i\omega$ are the roots of the characteristic equation (22).

The unknowns $\tilde{A}_{nk}$, $\tilde{B}_{nk}$ in (31) are expressed by

$$\begin{aligned}\tilde{A}_{nk}(\eta) &= C_k(\eta) B_{nk}(\eta),\\ \tilde{B}_{nk}(\eta) &= C_k(\eta) C_{nk}(\eta),\end{aligned} \tag{34}$$

where the constants $C_k(\eta)$ verify the recurrence relation

$$C_k(\eta) = \frac{\sqrt{k^2 + \eta^2}}{k(2k+1)} C_{k-1}(\eta), \tag{35}$$

with

$$C_0(\eta) = \frac{p\,|\,\Gamma(1 + i\eta)\,|}{2!}, \tag{36}$$

Γ is the gamma function and

$$|\frac{\Gamma(1+i\eta)}{\Gamma(2)}|^2=\prod_{n=0}[1+\frac{\eta^2}{(2+n)^2}]^{-1}. \tag{37}$$

The constants $A_m^{(k)}$, $B_m^{(k)}$ are related by the relation

$$B_m^{(k)}(\eta)=m\Delta A_m^{(k)}(\eta), \tag{38}$$

where $A_m^{(k)}(\eta)$ are given by

$$\begin{aligned} &A_{k+1}^{(k)}=1, \quad A_{k+2}^{(k)}=\frac{\eta}{k+1} \\ &(m+k)(m-k-1)A_m^{(k)}=2\eta A_{m-1}^{(k)}-A_{m-2}^{(k)}, \quad m>k+2 \end{aligned} \tag{39}$$

The constants $B_{nk}(\eta)$, $C_{nk}(\eta)$ depend on initial conditions and on coefficients from (17).

From (37) and $(32)_2$ we obtain

$$\psi_k(\mu t,\eta)=\sum_{m=k+1}(\mu t)^{m-k-1}B_m^{(k)}(\eta). \tag{40}$$

The function $\psi(\mu t,\eta)$ is linked to the $\Phi(\mu t,\eta)$ by

$$\Psi(\mu t,\eta)=\mu\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mu t,\eta)=\mu\dot{\Phi}(\mu t,\eta). \tag{41}$$

By noting

$$F_k(\mu t,\eta)=C_k(\eta)\,(\mu t)^{k+1}\Phi_k(\mu t,\eta) \tag{42}$$

the representation (31) becomes

$$A_n(t)=\sum_{\eta,k=0}\{B_{nk}(\eta)F_k(\mu t,\eta)+C_{nk}(\eta)\dot{F}_k(\mu t,\eta)\}]. \tag{43}$$

We have

$$F_0(\mu t,0)=\sin\mu t, \quad \dot{F}_0(\mu t,0)=\cos\mu t. \tag{44}$$

Finally, the LEM representation of the solution of the nonlinear system of equations (17) or (18) with the initial conditions (12) is found to be [10]

$$z_n(t) = A_n(t) = \sum_{\eta,k=0} \{B_{nk}(\eta)F_k(\mu t,\eta) + C_{nk}(\eta)\dot{F}_k(\mu t,\eta)\}] \tag{45}$$

where the functions $F_k(\mu t,\eta)$ have the form of Coulomb wave functions. For this, we call the functions $F_k(\mu t,\eta)$ the Coulomb vibrations.

As mentioned above, the solutions of nonlinear equations that govern the motion of a double pendulum are written as a linear superposition of Coulomb vibrations.

The first terms in (45) representing the linear part of the solution of (17) or (18) in the case of small oscillations ($n = 1,2,\ k = 0,1,\ \eta = 0,\ \omega = \delta = \tilde{\mu} = 0$) are

$$\begin{aligned} z_1 &= B_{10}(0)\sin p_1 t + C_{10}(0)\cos p_1 t + \\ &+ B_{11}(0)\sin p_2 t + C_{11}(0)\cos p_2 t, \\ z_2 &= B_{20}(0)\sin p_1 t + C_{20}(0)\cos p_1 t + \\ &+ B_{21}(0)\sin p_2 t + C_{21}(0)\cos p_2 t, \end{aligned} \tag{46}$$

$$\begin{aligned} B_{10} &= B_{20}\left(1 - \frac{p_1^2}{\gamma}\right), \quad C_{10} = C_{20}\left(1 - \frac{p_2^2}{\gamma}\right), \\ B_{11} &= B_{21}\left(1 - \frac{p_1^2}{\gamma}\right), \quad C_{11} = C_{21}\left(1 - \frac{p_2^2}{\gamma}\right), \end{aligned} \tag{47}$$

where p_1, p_2 are roots of the characteristic equation (22), and the constants B_{20}, B_{21}, C_{20} and C_{21} are determined from initial conditions

$$z_1(0) = z_{10}\ ,\ z_2(0) = z_{20}\ ,\ \dot{z}_1(0) = z_{30}\ ,\ \dot{z}_2(0) = z_{40}. \tag{48}$$

For $\eta = 0$ and $k > 2$ in the solutions appear additional terms of the form $\sin(ap_1 + bp_2)t$, $\cos(ap_1 + bp_2)t$ with $a + b = 1,2,3...$ For $\eta \neq 0$ the solutions will contain the functions $F[(ap_1 + bp_2)t, \eta]$ with $a + b = 1,2,3...$.

We add thar the distribution of stresses and displacements in a Love wave in a laminated half-space is analysed in [20] and the microwave ultrasonics in solid state physics in [21]. Details on the behavior of the one-dimensional Cantor composites can be found in [22].

4. RESULTS AND CONCLUSIONS

The undetermined system parameters are computed by using a genetic algorithm that exhibits a good convergence and accuracy. In all computations, the *measured* eigenfrequencies are computable from the direct problem. A *measurement noise* has been artificially introduced by multiplication of the data values by $1+r$, r being random numbers uniformly distributed in $[-\varepsilon, \varepsilon]$, with $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$.

Figures 2 and 3 show the displacements of the normal modes $\omega/2\pi$ = 242.8kHz and 475.6 kHz and respectively of the subharmonic modes $\omega/4\pi$ = 121.4 kHz and 237.8 kHz.

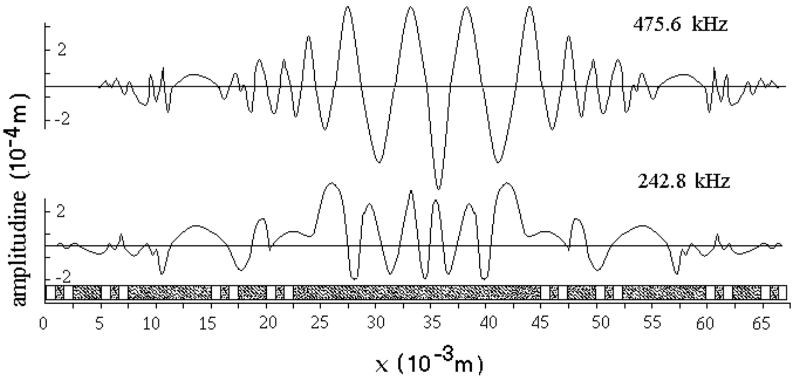

Fig. 2 – The amplitudes of displacement of the normal mode $\omega/2\pi = 475.6$ kHz and of the subharmonic mode $\omega/4\pi = 242.8$ kHz.

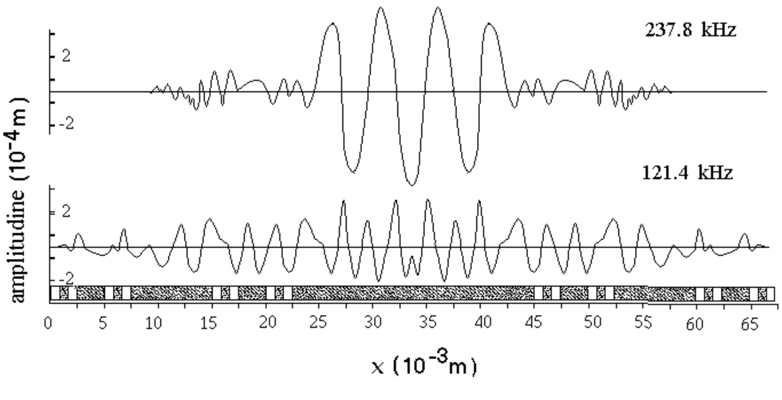

Fig. 3 – The amplitudes of the Love displacement of the normal mode $\omega/2\pi = 237.8$ kHz and of the subharmonic mode $\omega/4\pi = 121.4$ kHz..

Two kind of vibration regimes are found in out analysis, namely a localised-mode (fracton) regime represented in Fig. 4 for $\omega/2\pi$ = 2980.6 kHz, 3697.7 kHz and 4247.8 kHz and an extended-vibration(phonon) regime represented in Fig. 5 for $\omega/2\pi=$ 3355.2 kHz. A sketch of the plate geometry is given on the abscissa (dashed, ferrite and white, dielectric).

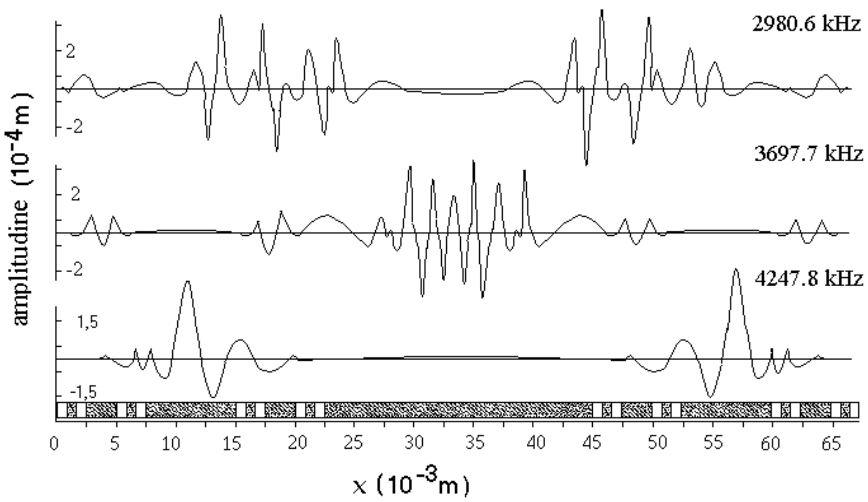

Fig. 4 – The normal amplitudes for three localised vibration modes ($\omega/2\pi=$ 2980.5 kHz, $\omega/2\pi=$ 3697.7 kHz and $\omega/2\pi=$ 4247.8 kHz).

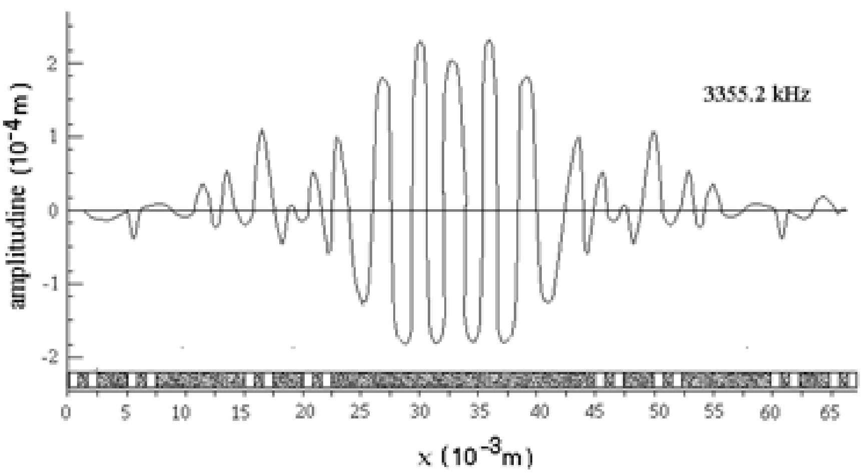

Fig. 5 – The extended-vibration(phonon) regime for $\omega/2\pi=$ 3355.2 kHz.

The fracton vibrations are mostly localised on a few elements, while the phonon vibrations essentially extend to the whole plate. For a homogeneous plate the difference $\omega_n - \omega/2$ is due to the symmetry of fundamental modes with respect to x. The ability of generating the subharmonics is given by the existence of a normal mode with small frequency difference $\omega_n - \omega/2$ and a large spatial overlap between the fundamental and the subharmonic displacement field.

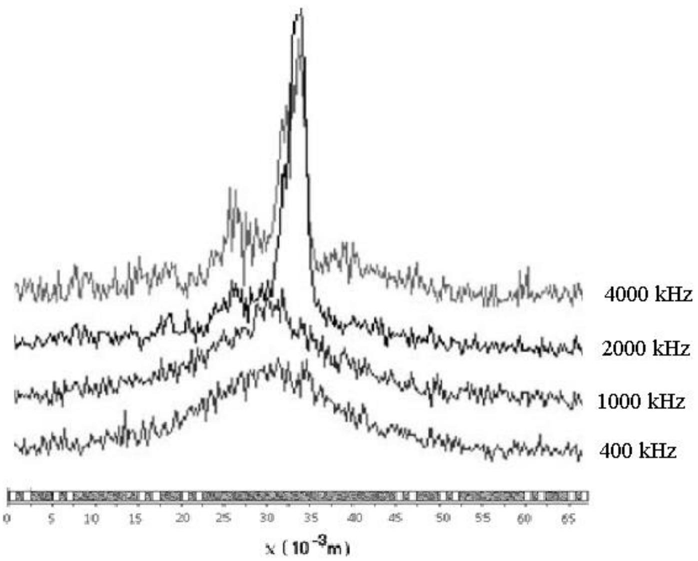

Fig. 6 – The chaotic tendency of the double pendulum for $\omega / 2\pi > 1000\,\text{kHz}$.

Starting the pendulum from a slightly different initial condition we obtain different results in a vastly different trajectory with chaotic behavior. Fig. 6 shows a chaotic behavior of the pendulum for $\omega / 2\pi > 1000\,\text{kHz}$.

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