SOME ASPECTS CONCERNING THE DISPLACEMENTS OF THE COMMON POINT FOR A PLANAR SYSTEM OF CANTILEVER BEAMS

NICOLAE-DORU STĂNESCU

Abstract. We consider a planar system of cantilever bars which has a common point and we want to determine the mechanical and geometrical characteristics of a new bar (situated in the same plane) in order to obtain a minimum or maximum value for a certain criterion in a certain set of values. The calculation combines the theory of screws to obtain the equations from which one can determine the displacement of the common point and the forces in bars, with the numerical methods to obtain the optimum value for some criterion, the optimum being considered in the previous sense. A numerical example highlights the theory.

Key words: screw theory, deformation, displacement, optimization, planar system.

1. INTRODUCTION

In a previous paper [1] the author studied a planar system of cantilever bars with one common end and obtained the position of equilibrium. The system is considered given and no variation of the geometrical or mechanical parameters is permitted. If one takes in consideration that the bars are spherically or cylindrically jointed [1, 2], then the calculation may simplify.

In this kind of systems the action upon the bars are considered known [3] for the simplicity of calculation, or the system of bars form a simple mechanism [4]. Different methods of study are presented [5] both for flexible systems [5] or Stewart platforms [6]. The Stewart platforms [7] may lead to different kind of absorbers [8], while other simple mechanisms lead to other suspended situations [9]. Authors focus their studies on the control of mechanisms [10] or to mechanisms that contain both rigid and elastic elements [11]. The dynamics of some simple mechanisms or Stewart platforms is also studied in the case of vibrations [12], or for transient motion [13].

National University of Science and Technology “Politehnica” of Bucharest, Pitești University Center

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Simple equations are obtained using the screw coordinates [14].

In their studies authors use the equations given by the theory of classical mechanics, theory of Lagrange, equations given by multibody theory (but only for small numbers of bodies because the mathematical apparatus becomes very difficult).

In [1] are described the main working hypotheses: no deformations in linkages, small deformations and displacements (in order to apply the classical theory of the strength of materials), bars of negligible weights, the forces and moments act at the common points, there is no error of manufacturing for the bars (their dimensions are nominal ones). Moreover, the force at the common point is situated in the plane of the bars, while the moment is along the perpendicular of the plane of bars.

As we will see, the calculation of the optimization can be performed, in the general case, only by numerical methods.

2. THE MECHANICAL SYSTEM

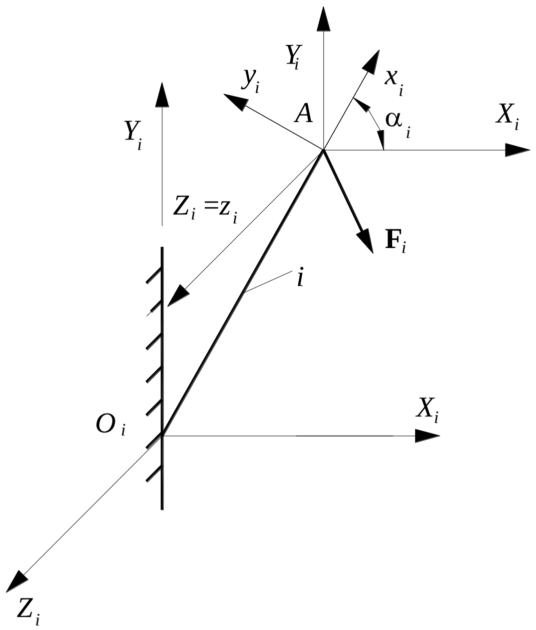

Fig. 1 – Mechanical system [1].

We consider a planar system of cantilever beams. From this system we consider only one beam for simplicity (Fig. 1). This bar is denoted by i; it has a length equal to l_i, a cross section area equal to A_i, the moments of inertia I_{z_i} and I_{y_i}, and the modulus of elasticity E_i. We attach a rectangular system of coordinates Ax_iy_i to the bar, such that Ax_i is along the bar (A is the common end of the bars) and Ax_i axis is in the exterior of the bar.

Let O be the other end of the bar. At this point we consider another system of coordinates such that $O_iX_iY_i$ has the axes parallel to the axes of a fixed reference frame (not represented in the figure). A system with the axes parallel to the axes of the system $O_iX_iY_i$ is also considered at the point A; this system is denoted by AX_iY_i. Let α_i be the angle between the AX_i-axis and Ax_i axis. The axes AZ_i, Az_i, and O_iZ_i are also drawn in figure, but they have no role in the future analysis. They are used only for the correct written of some certain matrices.

The system is acted by force $\mathbf{F}$ in the plane of bars (represented in figure) and moment $\mathbf{M}$ perpendicular to the plane of bars (not represented in figure).

Further on, we will use the notations given in reference [1].

The following matrices are defined:

– matrix of rigidity of the bare i

$$[\mathbf{k}_i]=\begin{bmatrix} 0 & \dfrac{E_iA_i}{l_i} & 0 \\ \dfrac{6E_iI_{z_i}}{l_i^2} & 0 & \dfrac{12E_iI_{z_i}}{l_i^3} \\ \dfrac{4E_iI_{z_i}}{l_i} & 0 & \dfrac{6E_iI_{z_i}}{l_i^2} \end{bmatrix}; \tag{1}$$

– matrix of flexibility of the bar i (which is the inverse of the matrix of rigidity of the bar i)

$$[\mathbf{h}_i]=\begin{bmatrix} 0 & \dfrac{l_i^2}{2E_iI_{z_i}} & \dfrac{l_i}{E_iI_{z_i}} \\ \dfrac{l_i}{E_iA_i} & 0 & 0 \\ 0 & \dfrac{l_i^3}{3E_iI_{z_i}} & \dfrac{l_i^2}{2E_iI_{z_i}} \end{bmatrix}; \tag{2}$$

– matrix of forces with respect to the reference system $Ax_iy_iz_i$

$$\{\mathbf{F}_i\}=\begin{bmatrix} F_{x_i} & F_{y_i} & M_{z_i} \end{bmatrix}^{\mathrm{T}}; \tag{3}$$

– matrix of deformations with respect to the reference system $Ax_iy_iz_i$

$$\{\mathbf{\Delta}_i\} = \begin{bmatrix}\theta_{z_i} & \Delta x_i & \Delta y_i\end{bmatrix}^{\mathrm{T}}; \tag{4}$$

– matrix of passing from the reference frame $AX_iY_iZ_i$ to the reference frame $Ax_iy_iz_i$

$$[\mathbf{A}_i] = \begin{bmatrix}\cos\alpha_{1i} & \cos\alpha_{2i} & \cos\alpha_{3i}\\ \cos\beta_{1i} & \cos\beta_{2i} & \cos\beta_{3i}\\ \cos\gamma_{1i} & \cos\gamma_{2i} & \cos\gamma_{3i}\end{bmatrix}. \tag{5}$$

Denoting with a star the parameters relative to the $A_iX_iY_iZ_i$ reference frame, and without star the parameters relative to the $A_ix_iy_iz_i$, we have [1]

– matrix of forces in the bar i

$$\{\mathbf{F}_1^*\} = \left[[\mathbf{I}] + [\mathbf{B}_2]^{-1}[\mathbf{B}_1] + \ldots + [\mathbf{B}_n]^{-1}[\mathbf{B}_1]\right]^{-1}\{\mathbf{F}^*\}; \tag{6}$$

– matrix of deformations

$$\{\mathbf{\Delta}^*\} = [\mathbf{A}_1]^{\mathrm{T}}[\mathbf{h}_1][\mathbf{G}_1]\{\mathbf{F}_1^*\}. \tag{7}$$

It results

$$\{\mathbf{F}_i^*\} = [\mathbf{B}_i]^{-1}[\mathbf{B}_1]\{\mathbf{F}^*\}, \tag{8}$$

$$\{\mathbf{F}_i\} = [\mathbf{G}_i]\{\mathbf{F}_i^*\}, \tag{9}$$

$$\{\mathbf{\Delta}_i\} = [\mathbf{h}_i]\{\mathbf{F}_i\}, \tag{10}$$

where

$$[\mathbf{G}_i] = \begin{bmatrix}\cos\alpha_i & \sin\alpha_i & 0\\ -\sin\alpha_i & \cos\alpha_i & 0\\ -\left(Y_A - Y_{O_i}\right) & X_A - X_{O_i} & 1\end{bmatrix}, \tag{11}$$

$$[\mathbf{B}_i] = [\mathbf{A}_i]^{\mathrm{T}}[\mathbf{h}_i][\mathbf{G}_i]. \tag{12}$$

3. OPTIMIZATION

Let us assume that someone wants to find the optimum (minimum or maximum) value of certain parameter relative to another one parameter p.

Let $[\mathbf{Q}]$ be a generic, arbitrary matrix with m lines and n columns,

$$[\mathbf{Q}]=\begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}; \tag{13}$$

in these conditions, we may write

$$\frac{\partial[\mathbf{Q}]}{\partial p}=\begin{bmatrix} \frac{\partial q_{11}}{\partial p} & \frac{\partial q_{12}}{\partial p} & \cdots & \frac{\partial q_{1n}}{\partial p} \\ \frac{\partial q_{21}}{\partial p} & \frac{\partial q_{22}}{\partial p} & \cdots & \frac{\partial q_{2n}}{\partial p} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial q_{m1}}{\partial p} & \frac{\partial q_{m2}}{\partial p} & \cdots & \frac{\partial q_{mn}}{\partial p} \end{bmatrix}. \tag{14}$$

It results

$$\frac{\partial[\mathbf{G}_i]}{\partial p}=\begin{bmatrix} -\sin\alpha_i\frac{\partial\alpha_i}{\partial p} & \cos\alpha_i\frac{\partial\alpha_i}{\partial p} & 0 \\ -\cos\alpha_i\frac{\partial\alpha_i}{\partial p} & -\sin\alpha_i\frac{\partial\alpha_i}{\partial p} & 0 \\ -\frac{\partial\left(Y_A-Y_{O_i}\right)}{\partial p} & \frac{\partial\left(X_A-X_{O_i}\right)}{\partial p} & 0 \end{bmatrix}, \tag{15}$$

$$\frac{\partial[\mathbf{A}_i]}{\partial p}=\begin{bmatrix} \sin\alpha_{1i}\frac{\partial\alpha_{1i}}{\partial p} & -\sin\alpha_{2i}\frac{\partial\alpha_{2i}}{\partial p} & -\sin\alpha_{3i}\frac{\partial\alpha_{3i}}{\partial p} \\ -\sin\beta_{1i}\frac{\partial\beta_{1i}}{\partial p} & -\sin\beta_{2i}\frac{\partial\beta_{2i}}{\partial p} & -\sin\beta_{3i}\frac{\partial\beta_{3i}}{\partial p} \\ -\sin\gamma_{1i}\frac{\partial\gamma_{1i}}{\partial p} & -\sin\gamma_{2i}\frac{\partial\gamma_{2i}}{\partial p} & -\sin\gamma_{3i}\frac{\partial\gamma_{3i}}{\partial p} \end{bmatrix}, \tag{16}$$

$$\frac{\partial\left[\left[\mathbf{A}_i\right]^{\mathrm{T}}\right]}{\partial p}=\left[\frac{\partial\left[\mathbf{A}_i\right]}{\partial p}\right]^{\mathrm{T}},$$

$$\frac{\partial\left[\mathbf{h}_i\right]}{\partial p}=\begin{bmatrix} 0 & \dfrac{\partial}{\partial p}\left(\dfrac{l_i^2}{2E_iI_{z_i}}\right) & \dfrac{\partial}{\partial p}\left(\dfrac{l_i}{E_iI_{z_i}}\right) \\ \dfrac{\partial}{\partial p}\left(\dfrac{l_i}{E_iA_i}\right) & 0 & 0 \\ 0 & \dfrac{\partial}{\partial p}\left(\dfrac{l_i^3}{3E_iI_{z_i}}\right) & \dfrac{\partial}{\partial p}\left(\dfrac{l_i^2}{2E_iI_{z_i}}\right) \end{bmatrix}, \tag{17}$$

$$\begin{aligned}\frac{\partial\left\{\mathbf{F}_1^*\right\}}{\partial p}=&-\left[\left[\mathbf{I}\right]+\left[\mathbf{B}_2\right]^{-1}\left[\mathbf{B}_1\right]+\ldots+\left[\mathbf{B}_n\right]^{-1}\left[\mathbf{B}_1\right]\right]^{-2}\\ &\times\frac{\partial\left[\left[\mathbf{I}\right]+\left[\mathbf{B}_2\right]^{-1}\left[\mathbf{B}_1\right]+\ldots+\left[\mathbf{B}_n\right]^{-1}\left[\mathbf{B}_1\right]\right]}{\partial p}\left\{\mathbf{F}^*\right\}+\\ &\left[\left[\mathbf{I}\right]+\left[\mathbf{B}_2\right]^{-1}\left[\mathbf{B}_1\right]+\ldots+\left[\mathbf{B}_n\right]^{-1}\left[\mathbf{B}_1\right]\right]^{-1}\frac{\partial\left\{\mathbf{F}^*\right\}}{\partial p},\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial\left\{\boldsymbol{\Delta}^*\right\}}{\partial p}=&\frac{\partial\left[\left[\mathbf{A}_1\right]^{\mathrm{T}}\right]}{\partial p}\left[\mathbf{h}_1\right]\left[\mathbf{G}_1\right]\left\{\mathbf{F}_1^*\right\}+\left[\mathbf{A}_1\right]^{\mathrm{T}}\frac{\partial\left[\mathbf{h}_1\right]}{\partial p}\left[\mathbf{G}_1\right]\left\{\mathbf{F}_1^*\right\}+\\ &\left[\mathbf{A}_1\right]^{\mathrm{T}}\left[\mathbf{h}_1\right]\frac{\partial\left[\mathbf{G}_1\right]}{\partial p}\left\{\mathbf{F}_1^*\right\}+\left[\mathbf{A}_1\right]^{\mathrm{T}}\left[\mathbf{h}_1\right]\left[\mathbf{G}_1\right]\frac{\partial\left\{\mathbf{F}_1^*\right\}}{\partial p},\end{aligned} \tag{19}$$

$$\frac{\partial\left[\mathbf{B}_i\right]}{\partial p}=\frac{\partial\left[\left[\mathbf{A}_i\right]^{\mathrm{T}}\right]}{\partial p}\left[\mathbf{h}_i\right]\left[\mathbf{G}_i\right]+\left[\mathbf{A}_i\right]^{\mathrm{T}}\frac{\partial\left[\mathbf{h}_i\right]}{\partial p}\left[\mathbf{G}_i\right]+\left[\mathbf{A}_i\right]^{\mathrm{T}}\left[\mathbf{h}_i\right]\frac{\partial\left[\mathbf{G}_i\right]}{\partial p}, \tag{20}$$

$$\begin{aligned}\frac{\partial\left\{\mathbf{F}_i^*\right\}}{\partial p}=&-\left[\mathbf{B}_i\right]^{-2}\frac{\partial\left[\mathbf{B}_i\right]}{\partial p}\left[\mathbf{B}_1\right]\left\{\mathbf{F}^*\right\}+\left[\mathbf{B}_i\right]^{-1}\frac{\partial\left[\mathbf{B}_1\right]}{\partial p}\left\{\mathbf{F}^*\right\}+\\ &\left[\mathbf{B}_i\right]^{-1}\left[\mathbf{B}_1\right]\frac{\partial\left\{\mathbf{F}^*\right\}}{\partial p},\end{aligned} \tag{21}$$

$$\frac{\partial[\mathbf{h}_i]}{\partial p}=\begin{bmatrix} 0 & \frac{\partial}{\partial p}\left(\frac{l_i^2}{2E_iI_{z_i}}\right) & \frac{\partial}{\partial p}\left(\frac{l_i}{E_iI_{z_i}}\right) \\ \frac{\partial}{\partial p}\left(\frac{l_i}{E_iA_i}\right) & 0 & 0 \\ 0 & \frac{\partial}{\partial p}\left(\frac{l_i^3}{3E_iI_{z_i}}\right) & \frac{\partial}{\partial p}\left(\frac{l_i^2}{2E_iI_{z_i}}\right) \end{bmatrix}. \tag{22}$$

As one may easily see, the partial derivatives have non-linear expressions; hence one has to solve a non-linear equation (in case when the optimization is required relative to a parameter). If two or more parameters are involved in the optimization process, then one has to solve a system of non-linear equations which is a very difficult task in the general case.

The previous comments lead us to the use of numerical methods in order to solve the non-linear equation, respectively the system of non-linear equations. Moreover, there is no guarantee that the value of p for which the extreme is found is a value that belongs to the interval admitted in the problem.

In these conditions, we will consider that the parameter p varies from the minimum value $p_{\min}$ to the maximum value $p_{\max}$ with a step value equal to Δp. The optimum value (minimum or maximum) for another parameter q is the minimum or maximum value in the set of values $q(p_1)=q(p_{\min})=q_1$, $q(p_2)=q(p_1+\Delta p)=q_2$, …, $q(p_{\max})$.

Moreover, instead of optimum value of the parameter q we consider the optimum value of the parameter q in the set of $q(p_{\min})$, …, $q(p_{\max})$.

We can state that the optimum value in the set of $q(p_{\min})$, …, $q(p_{\max})$ is not necessary the optimum value of the parameter q for $p\in[p_{\min}, p_{\max}]$. These two values of optimum are closer one to another for smaller value of the step Δp.

Similar statements can be written if the reader wants to apply the method of finite elements.

3. EXAMPLE

The situation with four bars is presented in reference [1]. Now, we introduce the fifth bar.

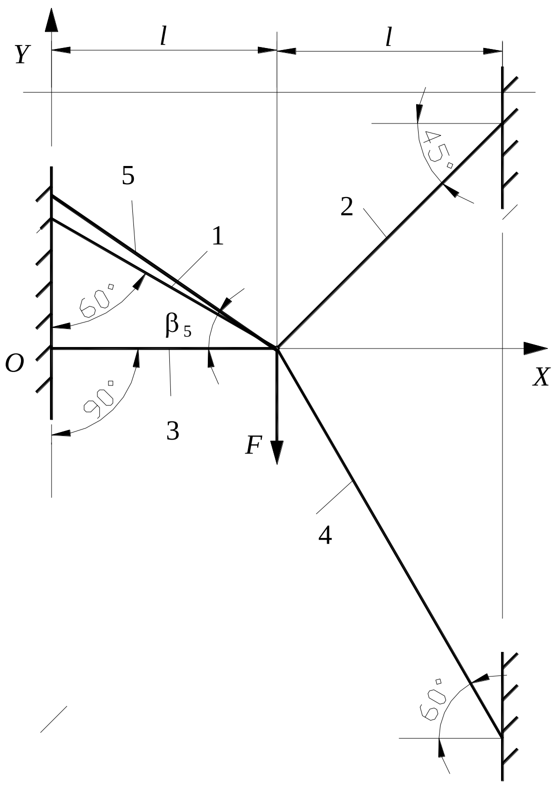

Fig. 1 – Example.

The following parameters characterize the standard situation: $l = 0.5\ \mathrm{m}$, $E_i = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, $d_i = 20\ \mathrm{mm}$, $i = \overline{1,4}$, $F = 1000\ \mathrm{N}$ (in the standard case the force acts vertically and has descendent sense; this is the situation for the first case and the second case presented further).

The fifth bar make the angle β_5 with the horizontal direction, where $\beta_{5\min} = \frac{\pi}{50}\ \mathrm{rad}$, and $\beta_{5\max} = \frac{\pi}{2} - \frac{\pi}{50}\ \mathrm{rad}$. The rest of the parameters are the same as for the first four bars.

One asks for the extremes of the linear displacement of the origin, the forces in the bars for the minimum and maximum displacements taking into account that the elastic modulus of a bar or the diameter of a certain bar can vary. The extremes are $E_{\min} = 10^{10}\ \mathrm{N/m^2}$, $E_{\max} = 3 \times 10^{11}\ \mathrm{N/m^2}$, the step being $\mathrm{d}E = 10^{10}\ \mathrm{N/m^2}$, and $d_{\min} = 15\ \mathrm{mm}$ and $d_{\max} = 25\ \mathrm{mm}$, the step being now $\mathrm{d}d = 1\ \mathrm{mm}$.

The first situation is characterized by variation of the modulus of elasticity E_1 from the minimum to the maximum value. The results are as follows:

– minimum displacement: $\Delta_{min} = 0.001944$ m for $\beta_5 = 86.39^{\,0}$ and $E_1 = 3\times 10^{11}$ N/m^2, while the forces in the bars are: for bar 1: $F_x = -14.09$ N, $F_y = -321.54$ N, $M_z = 82.42$ Nm; for bar 2: $F_x = -132.39$ N, $F_y = 21.52$ N, $M_z = -93.91$ Nm; for bar 3: $F_x = 30.52$ N, $F_y = -307.69$ N, $M_z = 76.93$ Nm; for bar 4: $F_x = 93.13$ N, $F_y = -84.35$ N, $M_z = -142.06$ Nm; for bar 5: $F_x = 22.83$ N, $F_y = -307.91$ N, $M_z = 76.62$ Nm;

– maximum displacement: $\Delta_{max} = 0.008651$ m for $\beta_5 = 3.61^{\,0}$ and $E_1 = 10^{10}$ N/m^2, while the forces in the bars read: for bar 1: $F_x = -15.29$ N, $F_y = -40.09$ N, $M_z = 12.23$ Nm; for bar 2: $F_x = -311.48$ N, $F_y = 373.41$ N, $M_z = -140.26$ N; for bar 3: $F_x = 152.83$ N, $F_y = -1369.70$ N, $M_z = 342.43$ Nm; for bar 4: $F_x = 173.99$ N, $F_y = 41.06$ N, $M_z = -215.75$ Nm; for bar 5: $F_x = -0.05$ N, $F_y = -4.72$ N, $M_z = 1.36$ Nm;

– minimum displacement in the x direction: $\Delta x_{min} = 0.0000002$ m for $\beta_5 = 86.40^{\,0}$ and $E_1 = 3\times 10^{11}$ N/m^2;

– maximum displacement in the x direction $\Delta x_{max} = 0.000001$ m for $\beta_5 = 32.20^{\,0}$ and $E_1 = 10^{10}$ N/m^2;

– minimum displacement in the y direction: $\Delta y_{min} = -0.008651$ m for $\beta_5 = 3.61^{\,0}$ and $E_1 = 10^{10}$ N/m^2;

– maximum displacement in the y direction $\Delta y_{max} = -0.001943$ m for $\beta_5 = 86.39^{\,0}$ and $E_1 = 3\times 10^{11}$ N/m^2;

– minimum rotation: $\theta_{min} = 0.000028^{\,0}$ for $\beta_5 = 3.62^{\,0}$ and $E_1 = 10^{10}$ N/m^2;

– maximum rotation: $\theta_{max} = 0.000081^{\,0}$ for $\beta_5 = 86.40^{\,0}$ and $E_1 = 3\times 10^{11}$ N/m^2.

The second situation is characterized by variation of the diameter of the first bar d_1 from the minimum to the maximum value. The results are given below:

– minimum displacement: $\Delta_{\min} = 0.001528\ \mathrm{m}$ for $\beta_5 = 86.40\ ^0$ and $d_1 = 25\ \mathrm{mm}$, while the forces in bars are: for bar 1: $F_x = -40.01\ \mathrm{N}$, $F_y = -419.83\ \mathrm{N}$, $M_z = 110.74\ \mathrm{Nm}$; for bar 2: $F_x = -118.38\ \mathrm{N}$, $F_y = 2.63\ \mathrm{N}$, $M_z = -88.13\ \mathrm{Nm}$; for bar 3: $F_x = 34.90\ \mathrm{N}$, $F_y = -241.89\ \mathrm{N}$, $M_z = 60.48\ \mathrm{Nm}$; for bar 4: $F_x = 91.48\ \mathrm{N}$, $F_y = -97.95\ \mathrm{N}$, $M_z = 60.48\ \mathrm{Nm}$; for bar 5: $F_x = 32.01\ \mathrm{N}$, $F_y == 294.95\ \mathrm{N}$, $M_z = 60.24\ \mathrm{Nm}$;

– maximum displacement: $\Delta_{\max} = 0.006495\ \mathrm{m}$ for $\beta_5 = 3.60\ ^0$ and $d_1 = 15[\mathrm{mm}]$, while the forces in the bars have the values: for bar 1: $F_x = -35.10\ \mathrm{N}$, $F_y = -223.79\ \mathrm{N}$, $M_z = 61.01\ \mathrm{Nm}$; for bar 2: $F_x = -248.55\ \mathrm{N}$, $F_y = 265.71\ \mathrm{N}$, $M_z = -119.99\ \mathrm{Nm}$; for bar 3: $F_x = 131.35\ \mathrm{N}$, $F_y = -1028.41\ \mathrm{N}$, $M_z = 257.11\ \mathrm{Nm}$; for bar 4: $F_x = 152.09\ \mathrm{N}$, $F_y = -6.33\ \mathrm{N}$, $M_z = -199.15\ \mathrm{Nm}$; for bar 5: $F_x = 0.20\ \mathrm{N}$, $F_y = -7.23\ \mathrm{N}$, $M_z = 1.01\ \mathrm{Nm}$;

– minimum displacement along the x direction: $\Delta x_{\min} = 0.0000003\ \mathrm{m}$ for $\beta_5 = 86.40\ ^0$ and $d_1 = 25\ \mathrm{mm}$;

– maximum displacement along x direction: $\Delta x_{\max} = 0.000001\ \mathrm{m}$ for $\beta_5 = 3.60\ ^0$ and $d_1 = 15\ \mathrm{mm}$;

– minimum displacement along y direction: $\Delta y_{\min} = -0.006495\ \mathrm{m}$ for $\beta_5 = 3.60\ ^0$ and $d_1 = 15\ \mathrm{mm}$;

– maximum displacement along y direction: $\Delta y_{\max} = -0.001528\ \mathrm{m}$ for $\beta_5 = 86.40\ ^0$ and $d_1 = 25\ \mathrm{mm}$;

– minimum rotation: $\theta_{\min} = 0.000046\ ^0$ for $\beta_5 = 3.66\ ^0$ and $d_1 = 15\ \mathrm{mm}$;

– maximum rotation: $\theta_{\max} = 0.000086\ ^0$ for $\beta_5 = 86.40\ ^0$ and $d_1 = 25\ \mathrm{mm}$.

The third situation is characterized by $F_x = 1000$ N (that is, the force has components both along the x direction and y direction) and by variation of the modulus of elasticity E_1 from the minimum to the maximum value. The results are as follows:

– minimum displacement: $\Delta_{min} = 0.001504$ mm for $\beta_5 = 66.67\ ^0$ and $E_1 = 3\times10^{11}\ \mathrm{N/m^2}$, while the forces in the bars are: for bar 1: $F_x = 283.27$ N, $F_y = -418.66$ N, $M_z = 63.78$ Nm; for bar 2: $F_x = 8.14$ N, $F_y = 127.26$ N, $M_z = 37.91$ Nm; for bar 3: $F_x = 339.11$ N, $F_y = -238.05$ N, $M_z = 59.52$ Nm; for bar 4: $F_x = 130.63$ N, $F_y = -166.77$ N, $M_z = -211.39$ Nm; for bar 5: $F_x = 238.84$ N, $F_y = -303.76$ N, $M_z = 50.19$ Nm;

– maximum displacement: $\Delta_{max} = 0.008514$ m for $\beta_5 = 3.61\ ^0$ and $E_1 = 10^{10}\ \mathrm{N/m^2}$, while the forces in bars read: for bar 1: $F_x = 5.15$ N, $F_y = -51.12$ N, $M_z = 12.04$ Nm; for bar 2: $F_x = -90.11$ N, $F_y = 584.07$ N, $M_z = 78.43$ Nm; for bar 3: $F_x = 788.62$ N, $F_y = -1348.07$ N, $M_z = 337.02$ Nm; for bar 4: $F_x = 296.20$ N, $F_y = -176.10$ N, $M_z = -428.79$ Nm; for bar 5: $F_x = 0.22$ N, $F_y = -8.79$ N, $M_z = 1.34$ Nm;

– minimum displacement along the x direction: $\Delta x_{min} = 0.000002$ m for $\beta_5 = 86.40\ ^0$ and $E_1 = 3\times10^{11}\ \mathrm{N/m^2}$;

– maximum displacement along the x direction: $\Delta x_{max} = 0.000006$ m for $\beta_5 = 3.61\ ^0$ and $E_1 = 10^{10}\ \mathrm{N/m^2}$;

– minimum displacement along the y direction: $\Delta y_{min} = -0.008514$ m for $\beta_5 = 3.61\ ^0$ and $E_1 = 10^{10}\ \mathrm{N/m^2}$;

– maximum displacement along the y direction: $\Delta y_{max} = -0.001504$ m for $\beta_5 = 66.67\ ^0$ and $E_1 = 3\times10^{11}\ \mathrm{N/m^2}$;

– minimum rotation: $\theta_{min} = 0.000039\ ^0$ for $\beta_5 = 3.62\ ^0$ and $E_1 = 10^{10}\ \mathrm{N/m^2}$;

– maximum rotation: $\theta_{max} = 0.000076\ ^0$ for $\beta_5 = 86.40\ ^0$ and $E_1 = 6\times10^{10}\ \mathrm{N/m^2}$.

The fourth situation is defined by $F_x = 1000$ N (that is, the force has components both along the x direction and y direction) and by variation of the diameter d_1 from the minimum to the maximum value. The results are given below:

– minimum displacement: $\Delta_{min} = 0.001139$ m for $\beta_5 = 60.92^0$ and $d_1 = 25$ mm, while the forces in bars are: for bar 1: $F_x = 288.94$ N, $F_y = -497.07$ N, $M_z = 82.56$ Nm; for bar 2: $F_x = 22.18$ N, $F_y = 112.42$ N, $M_z = 44.73$ Nm; for bar 3: $F_x = 339.50$ N, $F_y = -180.30$ N, $M_z = 45.08$ Nm; for bar 4: $F_x = 125.90$ N, $F_y = -173.02$ N, $M_z = -206.80$ Nm; for bar 5: $F_x = 223.47$ N, $F_y = -262.03$ N, $M_z = 34.44$ Nm;

– maximum displacement: $\Delta_{max} = 0.005739$ m for $\beta_5 = 3.60^0$ and $d_1 = 15$ mm, while the forces in bars are: for bar 1: $F_x = 164.83$ N, $F_y = -310.81$ N, $M_z = 53.91$ Nm; for bar 2: $F_x = -41.97$ N, $F_y = 412.48$ N, $M_z = 71.63$ Nm; for bar 3: $F_x = 639.84$ N, $F_y = -908.64$ N, $M_z = 227.16$ Nm; for bar 4: $F_x = 236.95$ N, $F_y = -183.31$ N, $M_z = -353.62$ Nm; for bar 5: $F_x = 0.39$ N, $F_y = -9.78$ N, $M_z = 0.90$ Nm;

– minimum displacement along the x direction: $\Delta x_{min} = 0.000002$ m for $\beta_5 = 86.40^0$ and $d_1 = 25$ mm;

– maximum displacement along the x direction: $\Delta x_{max} = 0.000005$ m for $\beta_5 = 3.60^0$ and $d_1 = 15$ mm;

– minimum displacement along the y direction: $\Delta y_{min} = -0.005739$ m for $\beta_5 = 3.60^0$ and $d_1 = 15$ mm;

– maximum displacement along the y direction: $\Delta y_{max} = -0.001139$ m for $\beta_5 = 60.92^0$ and $d_1 = 25$ mm;

– minimum rotation: $\theta_{min} = 0.000050^0$ for $\beta_5 = 3.63^0$ and $d_1 = 15$ mm;

– maximum rotation: $\theta_{\max} = 0.000075\,^{0}$ for $\beta_5 = 86.40\,^{0}$ and $d_1 = 25$ m.

The previous results lead us to the following conclusions:

– The minimum and maximum values for the displacement depend on which criterion is used, that is, some values are obtained if one wants the minimum or maximum values for the variations of the modulii of elasticity (the results are similar for the rest of bars), and some other values are obtained for the variations of the diameters of bars (the results are, again, similar for the rest of bars);

– The angles β_5 for which a certain optimization appears depends on what optimization is wanted;

– Similar conclusion may be written for the elasticity modulii or diameters of bars;

– Someone would expect to obtain minimum displacement (global, or along a certain direction, or rotation) for maximum modulus of elasticity or maximum diameter of a certain bar, but this is not a general conclusion. The minimum values depend on the configuration of the first bars, and the possibility of connection for the last bar (the angle β_5);

– A similar conclusion may be stated for maximum displacement, but now, these maximum values do not occur for minimum of the modulus of elasticity or minimum diameter of a certain bar;

– The maximum and minimum values of the displacement along a certain direction or about an axis perpendicular on the plane may have different signs, that is, there exists at least a position (an angle β_5) for which this displacement is equal to zero (vanishes);

– It is possible the maximum and minimum values of the displacement along a certain direction or about an axis perpendicular on the plane may have the same sign. In this case one may consider the minimum (maximum) value in modulus;

– The forces in bars are very different both in the same case and in different case. The values of these forces depend on the force **F** (the components F_x and F_y) and the structure of the mechanical system;

– The existence of the component F_x leads to other results of optimization; we can say that the structures of the new results are dramatically changed comparing to the original structure;

– One can see that the forces in certain bars have very small values (in modulus), while the forces in certain other bars have great values. Based on this observation, one may conclude that some bars have a negligible contribution and, consequently, they can be eliminated. This would be a great mistake because these small values of the forces are valid only for those particular positions (minimum or maximum displacements) and only for the possible orientation of the last bar.

4. CONCLUSIONS

In this paper we discussed some aspects concerning the optimization of the displacements (global, along one direction, or rotation) for a planar system of cantilever beams.

We proved that the analytical optimization may be used only in very particular cases; hence the general solution can be offered only by numerical calculations. Even if the equation (or the system of equations) is a simple one, there exists the possibility that the analytical solution does not belong to the interval (intervals) of variation for a parameter. In this case, the local optimum can be found only numerically.

It is clear that many parameters influence the results (e.g. the geometrical and mechanical structure of the mechanical system (the number of initial bars, their geometrical and mechanical characteristics; the number of bars we want to add as well their geometrical and mechanical characteristics), the force (and moment) that acts upon the mechanical system etc.).

The results cannot be obtained by an intuitive method, that is, for instance a greater modulus of elasticity or a greater diameter of a certain bar leads to smaller displacement of the origin, or smaller displacement along a certain direction, or a smaller rotation. Because of the constraints imposed to the position of the added bar, it is possible to obtain the optimum for other values of the modulus of elasticity or diameter.

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